HÖLDER CONTINUITY PROPERTY OF FILLED-IN JULIA SETS IN $\mathbb{C}^n$

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ABSTRACT. It is proved that the pluricomplex Green function of the filled-in Julia set J associated with a polynomial mapping in $\mathbb{C}^n$ is Hölder continuous. This yields in particular that J preserves Markov's inequality.

1. INTRODUCTION

Different classes of Julia sets have been already studied by several authors (see [CG] and the references given there). In this paper following M.Klimek [K1] we will consider the filled-in Julia sets defined as follows:

Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying

$$(1.1) \quad \liminf_{|z| \rightarrow \infty} \frac{|P(z)|}{|z|^\delta} > 0$$

with some $\delta > 1$. We define the *filled-in Julia set* associated with P to be the set

$$J_P := \{z \in \mathbb{C}^n : \{P^j(z)\}_{j \in \mathbb{N}_0} \text{ is bounded}\},$$

where P^j denotes the j -th iteration of the polynomial mapping P .

We are interested in potential-theoretic properties of these sets. Let us recall the definition of the L -extremal function in $\mathbb{C}^n$. For a compact subset E of $\mathbb{C}^n$ the function

$$V_E := \sup\{u : u(z) \leq \beta + \log^+ |z| \text{ for } z \in \mathbb{C}^n \text{ and } u \leq 0 \text{ on } E\}$$

is called the *L -extremal function corresponding to E* . A compact set E in $\mathbb{C}^n$ is said to be *L -regular* if V_E is continuous. It is well-known that V_E is a multidimensional counterpart of the classical Green function. For background material and thorough bibliography we refer the reader to [S2] or [K2].

In [K3] M.Klimek proved the following theorem, which answers the question about the continuity of the L -extremal function corresponding to a filled-in Julia set:

Theorem 1.1 ([K3, Corollary 6]). *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then the filled-in Julia set J_P associated with P is compact, polynomially convex and L -regular. Moreover*

$$(1.2) \quad P^{-1}(J_P) = J_P = P(J_P).$$

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In this paper we show that it actually is Hölder regular, i.e. the following theorem holds:

Theorem 1.2. *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then there exist constants $C > 0$ and $\alpha \in (0, 1]$ such that*

$$(HCP) \quad V_{J_P}(z) \leq C[\text{dist}(z, J_P)]^\alpha, \quad \text{if } \text{dist}(z, J_P) \leq 1.$$

In other words, Theorem 1.2 shows that the pluricomplex Green function V_{J_P} of the filled-in Julia set J_P has a Hölder continuity property.

The corresponding theorem in the case $n = 1$ is due to N. Sibony (1981). For the proof of that fact for quadratic polynomials we refer the reader to [CG, Theorem 8.3.2]. Our proof is based on a similar idea. This result was also already known for certain polynomial mappings on $\mathbb{C}^2$. J.E. Fornæss and N. Sibony studied properties of Julia sets associated with complex Hénon mappings (see [FS]). They obtained Hölder continuity of the L -extremal function corresponding to a set from this class (see [FS, Theorem 1.2]) and they mentioned that the proof easily extends to some bigger family of generalized Hénon mappings.

Recently A. Edigarian [E] found a nice generalization of Theorem 1.2; namely he proved that every L -regular compact set E in $\mathbb{C}^n$ has the Hölder continuity property, if there exist a C^∞ -mapping F and a $\delta > 1$ such that $V_E(F(z)) \geq \delta V_E(z)$ for $z \in \mathbb{C}^n$.

We say that a set E in $\mathbb{C}^n$ preserves *Markov's inequality* if for each polynomial p

$$(M) \quad |\text{grad } p(z)| \leq M(\deg p)^m \sup_E |p|, \quad \text{for } z \in E,$$

where M and m are positive constants depending only on E . Since A. A. Markov proved (M) for $E = [-1, 1] \subset \mathbb{C}$ in 1889 (see [M]), it has played an important role in the developing of the constructive theory of functions and has become the object of extensive research (see [RS] in the one-dimensional case, [PP] for $\mathbb{R}^n$ and [P] for $\mathbb{C}^n$).

It is well-known that the (HCP) property is sufficient for E to preserve Markov's inequality ([S1, Remark after Lemma 1]). Hence by Theorem 1.2 we immediately get

Corollary 1.3. *Let $P : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping satisfying (1.1) with some $\delta > 1$. Then the filled-in Julia set J_P associated with P satisfies Markov's inequality.*

2. PRELIMINARIES

From now on, P denotes a polynomial mapping of degree d satisfying (1.1) with some $\delta > 1$.

Most of the material in this paper relies on the following transformation rule for the L -extremal functions (see [K2, Theorem 5.3.1]):

Theorem 2.1 (M. Klimek). *Let E be a subset of $\mathbb{C}^n$. Then*

$$(2.1) \quad \frac{1}{d} (V_E \circ P) \leq V_{P^{-1}(E)} \leq \frac{1}{\delta} (V_E \circ P).$$

Let us mention the following trivial consequences of (2.1):

$$(2.2) \quad \frac{1}{d^j} (V_E \circ P^j) \leq V_{P^{-j}(E)} \leq \frac{1}{\delta^j} (V_E \circ P^j) \quad \text{for } j \in \mathbb{N}_0,$$

and of (1.2):

$$(2.3) \quad P^{-j}(J_P) = J_P = P^j(J_P) \quad \text{for } j \in \mathbb{N}_0.$$

Here $P^{-j}(E)$ denotes $(P^j)^{-1}(E)$.

The following consequence of (1.1) will be needed in Section 3.

Lemma 2.2 ([K1, Corollary 4]).

$$J_P = \mathbb{C}^n \setminus \{z \in \mathbb{C}^n : \lim_{j \rightarrow \infty} |P^j(z)| = \infty\}.$$

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.2. In view of Theorem 1.1 J_P is compact, thus we can fix such an $r > 1$ that $J_P \subset B_r := \{z \in \mathbb{C}^n : |z| \leq r\}$. Then by (2.3) we get

$$(3.1) \quad P^j(J_P) \subset B_r, \quad \text{if } j \in \mathbb{N}_0.$$

We define $A \geq \delta$ and $\alpha \in (0, 1]$ as follows:

$$A := \max\{\delta, \max_{|z| \leq 2r} |P'(z)|\} \quad \text{and} \quad \alpha := \log_A \delta.$$

If z is in J_P , we have $V_{J_P}(z) = \text{dist}(z, J_P) = 0$, and inequality (HCP) is obvious.

Now let z be in $\mathbb{C}^n \setminus J_P$ with $|z| < 2r$. Since J_P is compact, there exists a $z_0 \in J_P$ such that $|z - z_0| = \text{dist}(z, J_P) > 0$. If $w \in [z_0, z] := \{(1-t)z_0 + tz : t \in [0, 1]\}$ and $w \neq z_0$ then w does not belong to J_P .

We pick $k = k(z)$ in $\mathbb{N}$ to satisfy the following conditions:

$$(3.2) \quad |P^j(w)| \leq 2r \quad \text{for each } w \in [z_0, z] \text{ and } 0 \leq j \leq k-1,$$

and

$$(3.3) \quad \text{there exists a } z_1 \in [z_0, z] \text{ with } |P^k(z_1)| > 2r.$$

Note that we can find such a point according to Lemma 2.2.

Let $w \in [z_0, z]$ be given. By the definition of A we conclude that

$$\begin{aligned} |(P^k)'(w)| &= |P'(P^{k-1}(w)) \circ \dots \circ P'(P(w)) \circ P'(w)| \\ &\leq |P'(P^{k-1}(w))| \dots |P'(w)| \leq A^k \end{aligned}$$

since $|P^{k-1}(w)|, \dots, |P(z)|, |z| \leq 2r$ from (3.2). Hence integration along $[z_0, z_1]$ gives $|P^k(z_0) - P^k(z_1)| \leq A^k |z_0 - z_1|$. Combining this with (3.1) and (3.3) we obtain $2r < |P^k(z_1)| \leq A^k |z_0 - z_1| + |P^k(z_0)| \leq A^k \text{dist}(z, J_P) + r$, since $|z_0 - z_1| \leq |z_0 - z| = \text{dist}(z, J_P)$. This gives $A^k \text{dist}(z, J_P) \geq r > 1$ by the choice of r . Consequently by the definition of α we get

$$(3.4) \quad [\text{dist}(z, J_P)]^\alpha > (A^{-k})^\alpha = \delta^{-k}.$$

In view of (2.3) and (2.2) we obtain

$$(3.5) \quad V_{J_P}(z) = V_{P^{-k}(J_P)}(z) \leq \delta^{-k} V_{J_P}(P^k(z)).$$

We define now $C := \sup\{V_{J_P}(P(w)) : |w| \leq 2r\}$. Note that C is a finite number, because the polynomial mapping P is continuous and by Theorem 1.1 the L -extremal function V_{J_P} is also continuous. From (3.2) we get $|P^{k-1}(z)| \leq 2r$, whence $|V_{J_P}(P^k(z))| \leq C$. Thus in view of (3.4) and (3.5)

$$V_{J_P}(z) \leq C [\text{dist}(z, J_P)]^\alpha.$$

Moreover, if $\text{dist}(z, J_P) \leq 1$, then $|z| \leq \text{dist}(z, J_P) + |z_0| \leq 1 + r \leq 2r$ and this completes the proof of our assertion.

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