ON ZSIGMONDY PRIMES

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ABSTRACT. We present simple proofs of Walter Feit's results on large Zsigmondy primes.

We present simple proofs of known results related to Zsigmondy primes. We recall that if a, n are integers greater than 1, then a prime p is called a *Zsigmondy prime* for $\langle a, n \rangle$ if $p \nmid a$ and the order of $a \pmod{p}$ equals n (see [2], [4, §5], and Theorem 3 below). If p is a Zsigmondy prime for $\langle a, n \rangle$, then $n \mid p - 1$; thus $p \geq n + 1$. A Zsigmondy prime p for $\langle a, n \rangle$ is called a *large Zsigmondy prime* for $\langle a, n \rangle$ if $p > n + 1$ or p^2 divides $a^n - 1$ (equivalently, a prime p is a large Zsigmondy prime for $\langle a, n \rangle$ iff p is a Zsigmondy prime for $\langle a, n \rangle$ satisfying $|a^n - 1|_p > n + 1$. See [2]). Zsigmondy primes are used in finite group theory (see, e.g., [1]). For applications of large Zsigmondy primes to finite groups see [3] and [4].

The main results that we reprove here are Theorem 3 (Zsigmondy's Theorem) and Theorem 10 (due to Walter Feit).

We now recall some basic properties of cyclotomic polynomials, which we will use below (see, e.g., [6, Preliminaries, sec. 1]). For $n \geq 1$ the cyclotomic polynomial $\Phi_n(X)$ is defined as

$$\Phi_n(X) = \prod_{i=1}^{\varphi(n)} (X - \epsilon_i),$$

where $\epsilon_1, \dots, \epsilon_{\varphi(n)}$ are the primitive roots of unity of order n and $\varphi(n)$ is Euler's totient function. If $n > 1$, then $\Phi_n(a) > 0$ for all real a since $\Phi_2(X) = X + 1$, and for $n > 2$, $\Phi_n(X)$ is a monic polynomial over the integers with no real roots. Moreover, if $n > 1$ and $a > 1$ are integers, then $\Phi_n(a) > 1$ since $\Phi_n(a) = \prod_{i=1}^{\varphi(n)} |a - \epsilon_i| > (a - 1)^{\varphi(n)} \geq 1$.

Except for such basic facts, the paper is self-contained. Some of the proofs included here are the usual ones.

Lemma 1. *Let $a > 1$ and $n = q^i r$ be integers, where q is a prime, $i \geq 1$ and r is a positive integer not divisible by q . Let $b = a^{q^{i-1}}$. Then*

$$\Phi_n(a) > (b^{q-2}(b-1))^{\varphi(r)}.$$

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Proof. We have

$$\Phi_n(a) = \frac{\Phi_r(a^{q^i})}{\Phi_r(a^{q^{i-1}})} = \frac{\Phi_r(b^q)}{\Phi_r(b)} = \frac{\prod_{i=1}^{\varphi(r)} (b^q - \epsilon_i)}{\prod_{i=1}^{\varphi(r)} (b - \epsilon_i)} > \left(\frac{b^q - 1}{b - 1} \right)^{\varphi(r)},$$

where the ϵ_i 's are roots of unity. Since $b^q - 1 \geq b^{q-2}(b^2 - 1)$ the lemma follows. $\square$

Remark. In the notation of the previous lemma we have the inequality

$$\Phi_n(a) \geq \left(\frac{b^q + 1}{b + 1} \right)^{\varphi(r)}.$$

If q is a Zsigmondy prime for the pair $\langle a, n \rangle$, then q divides $a^n - 1 = \prod_{d|n} \Phi_d(a)$; thus q divides $\Phi_n(a)$. The next proposition characterizes the Zsigmondy primes among the prime factors of $\Phi_n(a)$.

Proposition 2 (cf. [5, Satz 1]). *Let $a > 1$ and $n > 1$ be integers. Let q be a prime factor of $\Phi_n(a)$. Then q is a non Zsigmondy prime for $\langle a, n \rangle$ iff q divides n . In this case q is the largest prime factor of n , and $n = q^i r$, where r is a positive integer dividing $q - 1$; moreover, q^2 does not divide $\Phi_n(a)$ unless $q = n = 2$.*

Thus, if there are no Zsigmondy primes for $\langle a, n \rangle$, then $\Phi_n(a)$ is a power of q ; if also $n > 2$ then $\Phi_n(a) = q$.

Proof. If $q \mid n$, then $a^{\frac{n}{q}} \equiv a^n \equiv 1 \pmod{q}$; thus q is not a Zsigmondy prime for $\langle a, n \rangle$ (actually, we have already proved this above).

Conversely, assume that q is not a Zsigmondy prime for $\langle a, n \rangle$. Then there exists a prime factor p of n such that $a^{n/p} \equiv 1 \pmod{q}$. Since $\Phi_n(X)$ divides $\frac{X^n - 1}{X^{n/p} - 1}$ we obtain, for $c = a^{n/p}$, that q divides $\frac{c^p - 1}{c - 1} = \sum_{i=0}^{p-1} c^i \equiv p \pmod{q}$. Thus $q = p$ divides n . Moreover, since $a^{n/p} \not\equiv 1 \pmod{q}$ for any prime factor $p \neq q$ of n , it follows that the order of $a \pmod{q}$ is of the form n/q^i for some $i \geq 1$, and so $r := n/q^i$ divides $q - 1$. Thus q is the largest prime factor of n .

Let $c = a^{n/q}$.

If $q > 2$ let $d = c - 1$; thus $q \mid d$. We have

$$\frac{a^n - 1}{a^{\frac{n}{q}} - 1} = \frac{c^q - 1}{c - 1} = \frac{(1 + d)^q - 1}{d} = q + \sum_{i=2}^{q-1} \binom{q}{i} d^{i-1} \equiv q \pmod{q^2}.$$

Thus $q^2 \nmid \Phi_n(a)$.

If $q = 2$ and $n > 2$, then n is a power of 2 and $n \geq 4$. We have $\frac{c^2 - 1}{c - 1} = c + 1$. Since a is odd and $\frac{n}{2}$ is even, we obtain $c = a^{n/2} \equiv 1 \pmod{4}$; thus $c + 1 \equiv 2 \pmod{4}$. Hence $4 \nmid \Phi_n(a)$ as claimed. $\square$

Theorem 3 (Zsigmondy's Theorem). *Let a and n be integers greater than 1. There exists a prime divisor q of $a^n - 1$ such that q does not divide $a^j - 1$ for all j , $0 < j < n$, except exactly in the following cases:*

- (1) $n = 2$, $a = 2^s - 1$, where $s \geq 2$.
- (2) $n = 6$, $a = 2$.

Proof. It is easy to verify that if one of the conditions (1) or (2) holds, then there is no q satisfying our requirements.

Assume that there is no prime q such that the order of a modulo q is n .

If $n = 2$, then by Proposition 2, $\Phi_n(a) = a + 1 = 2^s$ for some integer s , and case (1) holds.

Assume that $n > 2$. By Proposition 2, $\Phi_n(a) = q$, where q is the largest prime factor of n .

If $q = 2$, then $n = 2^s$ and $\Phi_n(a) = a^{2^{s-1}} + 1 > 2$, a contradiction.

Thus $q \geq 3$. Let $n = q^i r$, where r is an integer not divisible by q . Set $b = a^{q^{i-1}}$. By Lemma 1, $b^{q-2} < q$. Hence $q = 3$ and $b = 2, a = 2$ (indeed, $2^{q-2} < q$ implies $q = 3$. Thus $b < 3$, that is, $b = 2$). Since 7 is a Zsigmondy prime for $\langle 2, 3 \rangle$, we see that $n = qr > 3$. Since n divides $q(q-1) = 6$, we conclude that $n = 6$. $\square$

For a proof of Zsigmondy's Theorem in a stronger form see [6, (P1.7)]. For previous proofs see the references in [6], and especially [5]. For the present proof I have used a proof of Zsigmondy's Theorem based on [1] due to Yakov Berkovich and Gregory Freiman.

We now turn to the proof of Feit's results on Zsigmondy primes.

From Proposition 2 we obtain:

Corollary 4. *Let a, n be integers greater than 1. Let q be the largest prime factor of n . Assume that there are Zsigmondy primes for $\langle a, n \rangle$, but no large Zsigmondy primes. Then $n + 1$ is the unique Zsigmondy prime for $\langle a, n \rangle$. If $n > 2$, then either $\Phi_n(a) = n + 1$ or $\Phi_n(a) = q(n + 1)$.*

Corollary 4 shows that for $a > 1$ and $n > 1$, the largest prime factor of $a^n - 1$ is $\geq n + 1$. For a short review of far reaching generalizations of this remark see [7, Chapter 2, Section II.G]. Especially see [8-11].

Lemma 5. *For $n \geq 1, n \neq 6$ we have $2^{\varphi(n)} \geq n$; thus for all $n \geq 1$ we have $2^{\varphi(n)} \geq \frac{2}{3}n$.*

Proof. Let $n > 1$. First let n be odd. Then $2^{\varphi(n)} \equiv 1 \pmod{n}$. Thus $2^{\varphi(n)} = kn + 1$ for some $k \geq 1$. It follows that $2^{\varphi(n)} > n$. If $k = 1$, let $s = \varphi(n)$. The s numbers 2^i for $0 \leq i \leq s - 1$ are coprime with $n = \varphi(2^s - 1)$ and are between 1 and n . Since $1 \leq 2^s - 2 \leq n$ and $(2^s - 2, n) = 1$, we have $2^s - 2 = 2^i$ for some $0 \leq i \leq s - 1$. Hence $2^{s-1} - 1 = 2^{i-1}$; so, $2^{s-1} - 1 = 1$ and $n = 3$. It follows that $2^{\varphi(n)} > 3n$ holds for odd $n > 3$.

For an even $n > 1$, let $n = 2^i m$, where m is odd. If $m = 1$, then $2^{\varphi(n)} = 2^{2^{i-1}} \geq 2^i = n$. If $i = 1$ and $m > 3$, then $2^{\varphi(n)} = 2^{\varphi(m)} > 2m = n$. If $i > 1$ and $m \geq 3$, then $2^{\varphi(n)} = 2^{2^{i-1}\varphi(m)} \geq 2^{i\varphi(m)} \geq 2^{i+\varphi(m)} > 2^i m = n$ since a product of two integers greater than 1 is greater than or equal to their sum. $\square$

As seen from the proof of the previous lemma, we have equality $2^{\varphi(n)} = n$ iff $n = 2, 4$.

Theorem 6 [2, Theorem A]. *If a and n are integers greater than 1, then there exists a large Zsigmondy prime for $\langle a, n \rangle$ except exactly in the following cases:*

- (1) $n = 2$ and $a = 2^s 3^t - 1$ for some natural $s \geq 0$ and $t = 0, 1$ with $s \geq 2$ if $t = 0$.
- (2) $a = 2$ and $n = 4, 6, 10, 12$ or 18.
- (3) $a = 3$ and $n = 4$ or 6.
- (4) $\langle a, n \rangle = \langle 5, 6 \rangle$.

Proof. It is easy to show that in each case there are no large Zsigmondy primes for $\langle a, n \rangle$.

For the converse, by Theorem 3 we may assume that there are Zsigmondy primes for $\langle a, n \rangle$, but no large Zsigmondy primes. First let $n = 2$. As in [2], since the greatest common divisor of $a - 1$ and $a + 1$ is at most 2 it follows that any odd prime factor of $a + 1$ is a Zsigmondy prime and so it equals $n + 1 = 3$. Hence case (1) holds.

Now assume that $n > 2$. By Corollary 4, $\Phi_n(a)$ equals either $n + 1$ or $q(n + 1)$, and $n + 1$ is prime. Thus n is even.

◦ Let n be a power of 2: $n = 2^i, i \geq 2$.

If $\Phi_n(a) = n + 1$ then

$$a^{2^{i-1}} = 2^i.$$

Since $2^i \leq a^i \leq a^{2^{i-1}} = 2^i$, we obtain $a = 2$, $2^{i-1} = i$, and $i = 2$, that is $n = 4$.

If $\Phi_n(a) = 2(n + 1)$, then $a^{2^{i-1}} + 1 = 2(2^i + 1)$; thus $a^{2^{i-1}} = 2^{i+1} + 1$, and $a \geq 3$. If $i = 2$, then $a = 3$ and $n = 4$. If $i > 2$, then $a^{2^{i-1}} \geq 3^{i+1} > 2^{i+1} + 1$, a contradiction.

◦ Assume that n is not a power of 2. Let q be the largest prime factor of n and let $n = q^i r$, where r is an integer not divisible by q . Thus r is even. Set $b = a^{q^{i-1}}$.

• First assume that

$$\Phi_n(a) = q(n + 1).$$

By Lemma 1

$$(b^{q-2}(b-1))^{\varphi(r)} < \Phi_n(a) = q(rq^i + 1) \leq q(rq(b-1) + 1).$$

Divide by $b - 1$ to obtain

$$(1) \quad b^{(q-2)\varphi(r)} < q(rq + 1).$$

By Proposition 2, we have $q(rq + 1) \leq q((q-1)q + 1) < q^3$, and thus

$$(2) \quad b^{(q-2)\varphi(r)} < q^3.$$

If $q = 3$, then $r = 2$ by Proposition 2, and $b < 21$ by (1); thus $i = 1, 2$. Hence $n = 6, 18$. Since $q(n + 1) = \Phi_n(a) = b^2 - b + 1$, we obtain for $n = 6$ that $b = a = 5$ and for $n = 18$ that $b = 8, a = 2$.

Now assume that $q > 3$.

If $r > 2$, then $\varphi(r) \geq 2$. By (2)

$$(3) \quad b^{2(q-2)} < q^3.$$

By induction we easily obtain that $2^{2(q-2)} > q^3$ for any integer $q \geq 6$. Thus $q = 5$. Since the inequality (3) does not hold for $b = 3$ and $q = 5$, it follows that $b = 2, i = 1, n = 5r$. Since $r \mid q - 1$, we obtain that $r = 4$. Thus $n + 1 = 21$ is not prime, a contradiction.

Let $r = 2$.

We have

$$\Phi_n(a) = \Phi_{2q^i}(a) = \frac{b^q + 1}{b + 1}.$$

Hence

$$b + 1 \equiv b^q + 1 \equiv 0 \pmod{q}.$$

Thus $b \geq q - 1$. By (1)

$$(q - 1)^{q-2} < q(2q + 1).$$

But for $q \geq 5$ we have $(q - 1)^{q-2} \geq (q - 1)^3 > q(2q + 1)$, since $q^3 - 5q^2 + 2q - 1 > 0$, a contradiction.

- Assume that $\Phi_n(a) = n + 1$.

Similarly to (1) we have

$$(4) \quad (b^{q-2})^{\varphi(r)} < rq + 1.$$

Divide by $2^{\varphi(r)}$ and use Lemma 5 to obtain

$$(5) \quad \left(\frac{b^{q-2}}{2}\right)^{\varphi(r)} < \frac{3}{2}q + 1.$$

Hence

$$(6) \quad b^{q-2} < 3q + 2.$$

For any integer $q \geq 7$ we show by induction that $2^{q-2} \geq 3q + 2$. Thus $q = 3, 5$.

Let $q = 3$.

Let $n = 2^j 3^i$. Since $r \neq 6$, by Lemma 5 we have $b^{q-2} < 2q + 2$ (cf. (6)), that is, $b < 8$. Hence $i = 1$. We have

$$\Phi_n(a) = \frac{\Phi_{2^j}(a^3)}{\Phi_{2^j}(a)} = \frac{c^3 + 1}{c + 1},$$

where $c = a^{2^{j-1}}$. Thus $c^2 - c + 1 = n + 1$, that is, $c(c - 1) = 2^j \cdot 3$. So $c = 3$ or $c - 1 = 3$. If $c = 3$ then $a = 3$, $n = 6$. If $c - 1 = 3$ then $4 = c = 2^j$, $j = 2$, $a = 2$, $n = 12$.

Let $q = 5$.

By (5) we obtain $2^{2\varphi(r)} < \frac{3}{2} \cdot 5 + 1$. Thus $4^{\varphi(r)} < 9$, which implies $\varphi(r) = 1$, $r = 2$. By (6), we have $b^3 < 17$, and so $b = a = 2$, $i = 1$, and $n = 10$. $\square$

In the sequel we prove [2, Theorem B], but first some auxiliary results.

Lemma 7. *Let $a > 1$ and $n > 2$ be integers. Then*

$$\Phi_n(a) > a^{\varphi(n)/2}.$$

Proof. If n is a power of 2, $n = 2^m$, then

$$\Phi_n(a) = a^{2^{m-1}} + 1 > a^{2^{m-1}} = a^{\varphi(n)}.$$

Otherwise, let q be a prime odd factor of n . Set $n = q^i r$, where r is an integer not divisible by q . Set $b = a^{q^{i-1}}$. By Lemma 1, we have

$$\Phi_n(a) > (b^{q-2})^{\varphi(r)} \geq \left(b^{\frac{q-1}{2}}\right)^{\varphi(r)} = a^{\varphi(n)/2}.$$

$\square$

Lemma 8 [2, Lemma 2.5]. *For $n \geq 1$ we have $\varphi(n) \geq \sqrt{n}/2$.*

Proof. If p is an odd prime and $m \geq 1$ then $\varphi(p^m) = (p-1)p^{m-1} \geq \sqrt{p^m}$. Also $\varphi(2^m) = 2^{m-1} \geq \sqrt{2^m}/2$. Since φ is a multiplicative function ($\varphi(mn) = \varphi(m)\varphi(n)$ for m, n coprime) the lemma follows. $\square$

As a result of Lemmas 7 and 8 we obtain:

Corollary 9. *For any integers $a > 1, n > 2$ we have*

$$\Phi_n(a) > a^{\frac{\sqrt{n}}{4}}.$$

Theorem 10 [2, Theorem B]. *Let N be a positive integer. Then for all but finitely many pairs of integers $\langle a, n \rangle$ with $a > 1$ and $n > 2$, there exists a Zsigmondy prime p with $|a^n - 1|_p > nN + 1$.*

Proof. Let $a > 1$ and $n > 2$ be positive integers such that there are no Zsigmondy primes p for $\langle a, n \rangle$ with $|a^n - 1|_p > nN + 1$. Since any Zsigmondy prime p satisfies $p \equiv 1 \pmod{n}$, there are at most N Zsigmondy primes satisfying $p \leq nN + 1$. Let q be the largest prime factor of n . Hence, by Proposition 2

$$\Phi_n(a) \leq q(nN + 1)^N.$$

By Corollary 9 this implies $a^{\frac{\sqrt{n}}{4}} < n(nN + 1)^N$. The theorem follows. $\square$

Finally, we recall that Zsigmondy's Theorem was used by Wedderburn in order to prove that any finite division ring is commutative [12]. We reproduce here one of the Wedderburn's proofs slightly revised. Let D be a finite division ring of dimension n over its prime subfield $\mathbb{F}_p$. First assume that there is a Zsigmondy prime q for $\langle p, n \rangle$. Let $g \in D \setminus \{0\}$ be an element of order q . Let F be the subring of D generated by g . Let $m = [F : \mathbb{F}_p]$. Since $g^{p^m - 1} = 1$ we have $q \mid p^m - 1$. Thus $m = n$ and so $D = F$ is commutative.

If there are no Zsigmondy primes for $\langle p, n \rangle$, then, by Zsigmondy's Theorem, either $n = 2$, or $n = 6$ and $p = 2$. If $n = 2$, then D is commutative since it is the subring generated by any element in $D \setminus \mathbb{F}_p$. If $n = 6$ and $p = 2$, then the order of 2 (mod 9) is 6. Since $D \setminus \{0\}$ contains a subgroup of order 9 (which is abelian), we can use the previous argument to complete the proof.

To obtain a uniform formulation of the proof for $n > 2$ note that Zsigmondy's Theorem implies that for any $n > 2$ and $a > 1$ there exists a prime p such that for $m = p$ or $m = p^2$, the order of a modulo m equals n .

Of course, there are simpler proofs of Wedderburn's Theorem. However, it is interesting that Wedderburn's Theorem is a simple consequence of Zsigmondy's Theorem.

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