LIFTING OF GENERATING SUBGROUPS

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ABSTRACT. Let $\varphi: G \rightarrow H$ be an epimorphism of finite groups. Suppose that G is generated by its subgroups $G_1, \dots, G_n$ and that H is generated by its subgroups $H_1, \dots, H_n$. Furthermore, suppose that $\varphi(G_i)$ and H_i are conjugate, $i = 1, \dots, n$. We prove that there exist $g_1, \dots, g_n \in G$ such that $G_1^{g_1}, \dots, G_n^{g_n}$ generate G and $\varphi(G_i^{g_i}) = H_i$, $i = 1, \dots, n$.

In this note we prove the following lifting principle for generating subgroups:

Main Theorem. *Let $\varphi: G \rightarrow H$ be an epimorphism of finite groups and let $G_1, \dots, G_n$ and $H_1, \dots, H_n$ be subgroups of G and H , respectively. Suppose that $G = \langle G_1, \dots, G_n \rangle$ and $H = \langle H_1, \dots, H_n \rangle$. Moreover, suppose that φ maps some conjugate of G_i onto H_i , $i = 1, \dots, n$. Then there exist conjugates $G_1^{g_1}, \dots, G_n^{g_n}$ of $G_1, \dots, G_n$ which are mapped by φ onto $H_1, \dots, H_n$, respectively, and which generate G .*

This theorem and its proof are motivated by the following result:

Theorem (Gaschütz [4]). *Let $\varphi: G \rightarrow H$ be an epimorphism of finite groups and let $\{h_1, \dots, h_n\}$ be a set of generators for H . Suppose that G can be generated by n elements. Then there exists a set $\{g_1, \dots, g_n\}$ of generators for G such that $\varphi(g_i) = h_i$, $i = 1, \dots, n$.*

Gaschütz proved this fact by means of a sophisticated counting argument. His proof has later been simplified by Roquette [2, Lemma 15.30]. This theorem had an important and quite surprising role in the model theory of fields [3]. See also [1] for a generalization of Gaschütz' theorem, which however does not cover our main theorem.

Proof of the Main Theorem. Let J be the collection of all subgroups F of G such that $\varphi(F) = H$. For $F \in J$, $h \in H$ and a left coset $gF \in G/F$ (where $g \in G$) let

$$A_i(F, h, gF) = gF \cap \varphi^{-1}(N_H(\varphi(G_i))h), \quad i = 1, \dots, n,$$

where N_H denotes the normalizer of a subgroup of H . Thus $g' \in A_i(F, h, gF)$ if and only if g and g' belong to the same left coset in G/F and $\varphi(G_i^{g'}) = \varphi(G_i)^h$. We show that the size of the finite set $A_i(F, h, gF)$ does not depend on the choice of h . Indeed, for any other $h' \in H$ we can take $f, f' \in F$ with $\varphi(f) = h$ and $\varphi(f') = h'$. Then $A_i(F, h', gF) = A_i(F, h, gF)f^{-1}f'$.

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For $F \in J$, for $h_1, \dots, h_n \in H$, and for left cosets $g_1F, \dots, g_nF \in G/F$ (where $g_1, \dots, g_n \in G$) set:

$$\begin{aligned} B(F, h_1, \dots, h_n, g_1F, \dots, g_nF) \\ = \left\{ (g'_1, \dots, g'_n) \in \prod_{i=1}^n A_i(F, h_i, g_iF) \mid F = \langle G_1^{g'_1}, \dots, G_n^{g'_n} \rangle \right\}. \end{aligned}$$

Claim. Let $F \in J$, let $h_1, \dots, h_n \in H$ satisfy $H = \langle \varphi(G_1)^{h_1}, \dots, \varphi(G_n)^{h_n} \rangle$, and let $g_i \in G$ satisfy $G_i^{g_i} \leq F$, $i = 1, \dots, n$. Then:

$$\begin{aligned} \prod_{i=1}^n A_i(F, h_i, g_iF) = \bigcup \left\{ B(F^*, h_1, \dots, h_n, g'_1F^*, \dots, g'_nF^*) \mid F^* \in J, \right. \\ \left. F^* \leq F, g'_iF = g_iF, i = 1, \dots, n \right\}. \end{aligned}$$

Indeed, it is straightforward to check that the right-hand side is contained in the left-hand side. Conversely, suppose that $(g'_1, \dots, g'_n) \in \prod_{i=1}^n A_i(F, h_i, g_iF)$ and set $F^* = \langle G_1^{g'_1}, \dots, G_n^{g'_n} \rangle$. Clearly, $(g'_1, \dots, g'_n) \in B(F^*, h_1, \dots, h_n, g'_1F^*, \dots, g'_nF^*)$. In addition, $g'_iF = g_iF$ and $G_i^{g_i} \leq F$, so one has $G_i^{g'_i} \leq F$ for each $1 \leq i \leq n$, whence $F^* \leq F$. Also $g'_i \in A_i(F, h_i, g_iF)$ implies that $\varphi(G_i^{g'_i}) = \varphi(G_i)^{h_i}$, and we obtain:

$$\varphi(F^*) = \langle \varphi(G_1^{g'_1}), \dots, \varphi(G_n^{g'_n}) \rangle = \langle \varphi(G_1)^{h_1}, \dots, \varphi(G_n)^{h_n} \rangle = H.$$

Therefore $F^* \in J$. Conclude that $(g'_1, \dots, g'_n)$ belongs to the right-hand side.

To see that this union is disjoint we need to prove that an n -tuple

$$(g''_1, \dots, g''_n) \in B(F^*, h_1, \dots, h_n, g'_1F^*, \dots, g'_nF^*)$$

already determines F^* and $g'_1F^*, \dots, g'_nF^*$. Now $F^* = \langle G_1^{g''_1}, \dots, G_n^{g''_n} \rangle$, and as $g''_i \in A_i(F^*, h_i, g'_iF^*)$ we have $g'_iF^* = g''_iF^*$, $i = 1, \dots, n$. This proves the claim.

By the assumption of the theorem there exist $h_1, \dots, h_n \in H$ with $\varphi(G_i)^{h_i} = H_i$, $i = 1, \dots, n$. Therefore

$$H = \langle \varphi(G_1)^{h_1}, \dots, \varphi(G_n)^{h_n} \rangle$$

as well as

$$H = \varphi(G) = \langle \varphi(G_1), \dots, \varphi(G_n) \rangle.$$

By the first part of the proof, $|A_i(F, h_i, g_iF)| = |A_i(F, 1, g_iF)|$ for every $F \in J$ and cosets $g_iF \in G/F$, $i = 1, \dots, n$. Assuming in addition that $G_i^{g_i} \leq F$ for each i , the claim implies that

$$\begin{aligned} & |B(F, h_1, \dots, h_n, g_1F, \dots, g_nF)| + \sum \left| B(F^*, h_1, \dots, h_n, g'_1F^*, \dots, g'_nF^*) \right| \\ &= \prod_{i=1}^n |A_i(F, h_i, g_iF)| = \prod_{i=1}^n |A_i(F, 1, g_iF)| \\ &= |B(F, 1, \dots, 1, g_1F, \dots, g_nF)| + \sum \left| B(F^*, 1, \dots, 1, g'_1F^*, \dots, g'_nF^*) \right|, \end{aligned}$$

where the sums are taken over all *proper* subgroups F^* of F which belong to J and over all left cosets $g'_1F^*, \dots, g'_nF^*$ in G/F^* satisfying $g'_iF = g_iF$, $i = 1, \dots, n$.

Note that when $g'_i F = g_i F$ one has $G_i^{g'_i} \leq F$ as well, $i = 1, \dots, n$. By induction on the size of $F \in J$ we now obtain that

$$|B(F, h_1, \dots, h_n, g_1 F, \dots, g_n F)| = |B(F, 1, \dots, 1, g_1 F, \dots, g_n F)|$$

for every $F \in J$ and $g_i F \in G/F$ which satisfy $G_i^{g_i} \leq F$, $i = 1, \dots, n$. Since $(1, \dots, 1) \in B(G, 1, \dots, 1, G, \dots, G)$ also $B(G, h_1, \dots, h_n, G, \dots, G) \neq \emptyset$. But for every n -tuple $(g'_1, \dots, g'_n)$ in the latter set we have $G = \langle G_1^{g'_1}, \dots, G_n^{g'_n} \rangle$ as well as

$$\varphi(G_i^{g'_i}) = \varphi(G_i)^{h_i} = H_i,$$

$i = 1, \dots, n$. This gives conjugates of $G_1, \dots, G_n$ as required. $\square$

REFERENCES

1. I. Efrat, *The elementary theory of free pseudo p -adically closed fields of finite corank*, J. Sym. Logic **56** (1991), 484–496. MR **93a**:03036
2. M. Fried and M. Jarden, *Field Arithmetic*, Springer, Heidelberg, 1986. MR **89b**:12010
3. M. Jarden and U. Kiehne, *The elementary theory of algebraic fields of finite corank*, Inventiones Math. **30** (1975), 275–294. MR **55**:8012
4. W. Gaschütz, *Zu einem von B.H. und H. Neumann gestellten Problem*, Math. Nachr. **14** (1956), 249–252. MR **18**:790b

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