CONSTRUCTION OF INVARIANT CURVES FOR SINGULAR HOLOMORPHIC VECTOR FIELDS

J. CANO

(Communicated by Mary Rees)

ABSTRACT. Camacho and Sad proved the existence of invariant analytic curves for germs of singular holomorphic foliations $\mathcal{F}$ over a two dimensional complex analytic variety M . Their proof is only of existential nature. Here we provide a simple constructive proof by giving criteria to choose a singular point at each blowing-up that follows an analytic invariant curve.

Our algorithm is founded on the stability by blowing-up of the property $(\star)$ introduced in the following definition.

Definition. Consider a singular holomorphic foliation $\mathcal{F}$ over a two dimensional complex analytic variety M , a normal crossings divisor E over M and a point $q \in E$. We say that the triple $(\mathcal{F}, E, q)$ has the property $(\star)$ if and only if one of the following properties holds:

- $(\star)$ -1: The point q lies exactly in one irreducible component S of E , which is invariant for $\mathcal{F}$ and the index $i_q(\mathcal{F}, S) \notin \mathbb{Q}_{(\geq 0)} = \{r \in \mathbb{Q}; r \geq 0\}$.
- $(\star)$ -2: The point q lies in two irreducible components S_+ and S_- of E (call this point a “corner”), both are invariant curves and there is a real number $a > 0$ such that:

$$\begin{aligned} i_q(\mathcal{F}, S_+) &\in \mathbb{Q}_{(\leq -a)} = \{r \in \mathbb{Q}; r \leq -a\}, \\ i_q(\mathcal{F}, S_-) &\notin \mathbb{Q}_{(\geq -1/a)} = \{r \in \mathbb{Q}; r \geq -1/a\}. \end{aligned}$$

- $(\star)$ -3: The point q lies exactly in one irreducible component S of E , it is a nonsingular point of $\mathcal{F}$ and S is transversal to $\mathcal{F}$ at q .

(The definition and basic properties of the index can be found in [1]).

Remark. If we have the property $(\star)$ -2, then q is not a simple (irreducible) singularity. If we have either $(\star)$ -1 and q is a simple singularity or $(\star)$ -3, then there is a nonsingular analytic invariant curve Γ through q transversal to E .

Theorem. Assume that $(\mathcal{F}, E, q)$ satisfies either $(\star)$ -1 or $(\star)$ -2. Consider the blowing-up $\pi : M' \rightarrow M$ at the point q . Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ by π . Put $D = \pi^{-1}(q)$ and $E' = \pi^{-1}(E)$. Then there is a point $q' \in D$ such that $(\mathcal{F}', E', q')$ satisfies the property $(\star)$.

Received by the editors October 24, 1995 and, in revised form, March 19, 1996.
 1991 *Mathematics Subject Classification.* Primary 34A05; Secondary 32S65.
 This work was supported by the I.A.S. under NSF grant # DMS-9304580.

Proof. If π is a dicritical blowing-up we immediately get a point $q' \in D$ such that $(\mathcal{F}', E', q')$ satisfies the property $(\star)$ -3.

Assume that π is non dicritical and hence D is an invariant curve for $\mathcal{F}'$.

Consider first the case that $(\mathcal{F}, E, q)$ satisfies $(\star)$ -1. Let S' be the strict transform of S by π and put $\{q'\} = D \cap S'$. Let $p_1, \dots, p_s$ be the singularities of $\mathcal{F}'$ on $D \setminus \{q'\}$. Suppose that $(\mathcal{F}', E', p_i)$ does not satisfy $(\star)$ -1 for any $i = 1, \dots, s$. Then we have that

$$i_{q'}(\mathcal{F}', D) = -1 - \sum_{i=1}^s i_{p_i}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -1)}.$$

Since $i_{q'}(\mathcal{F}', S') = i_q(\mathcal{F}, S) - 1 \notin \mathbb{Q}_{(\geq -1)}$, then $(\mathcal{F}', E', q')$ satisfies $(\star)$ -2.

Consider now the case that $(\mathcal{F}, E, q)$ satisfies $(\star)$ -2. Let S'_+ and S'_- be the strict transforms by π of S_+ and S_- respectively. Let $p_1, \dots, p_s$ be the singularities of $\mathcal{F}'$ on $D \setminus \{q_+, q_-\}$, where $q_+ = D \cap S'_+$ and $q_- = D \cap S'_-$. If $(\mathcal{F}', E', p_i)$ does not satisfy $(\star)$ -1 for $i = 1, \dots, s$, then

$$(1) \quad i_{q_+}(\mathcal{F}', D) + i_{q_-}(\mathcal{F}', D) = -1 - \sum_{i=1}^s i_{p_i}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -1)}.$$

Since $(\mathcal{F}, E, q)$ satisfies the property $(\star)$ -2 we have that

$$\begin{aligned} i_{q_+}(\mathcal{F}', S'_+) &= i_q(\mathcal{F}, S_+) - 1 \in \mathbb{Q}_{(\leq -(a+1))}, \\ i_{q_-}(\mathcal{F}', S'_-) &= i_q(\mathcal{F}, S_-) - 1 \notin \mathbb{Q}_{(\geq -\frac{a+1}{a})}. \end{aligned}$$

If $(\mathcal{F}', E', q_+)$ does not satisfy $(\star)$ -2 then $i_{q_+}(\mathcal{F}', D) \in \mathbb{Q}_{(\geq -\frac{1}{a+1})}$. By (1) $i_{q_-}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -\frac{a}{a+1})}$ and thus $(\mathcal{F}', E', q_-)$ satisfies $(\star)$ -2. $\square$

To get an analytic invariant curve Γ for $\mathcal{F}$ at q we proceed as follows. After the blowing-up with center q we take a point p_1 in the exceptional divisor E_1 with the property $(\star)$: if the blowing-up is dicritical we get $(\star)$ -3 and the algorithm stops, otherwise we get $(\star)$ -1 since the sum of the indices is -1 . Repeat. By reduction of singularities, in a finite number of steps we get either $(\star)$ -3 or an irreducible singularity satisfying $(\star)$. This gives an analytic invariant curve Γ' transversal to the divisor that projects over Γ .

REFERENCES

- [1] C. Camacho and P. Sad. *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math., **115**, (1982) 579-595. MR **83m**:58062

FACULTAD DE CIENCIAS, UNIVERSIDAD DE VALLADOLID, 47005-VALLADOLID, SPAIN
E-mail address: jcano@cpd.uva.es