MEHLER-FOCK TRANSFORMS OF GENERALIZED FUNCTIONS VIA THE METHOD OF ADJOINTS

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ABSTRACT. In this paper we analyse the Mehler-Fock transform of generalized functions via the method of adjoints. For a distribution of compact support, we prove that its Mehler-Fock transform agrees with its transform via the kernel method. A Paley-Wiener type theorem is established.

1. INTRODUCTION

The Mehler-Fock transform of a distribution f of compact support on I (from now on $I = (0, \infty)$) was defined by the kernel method in [10] as:

$$(1.1) \quad F(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}(\cosh x) \rangle, \quad \tau > 0,$$

where, as it is usual, $P_{-\frac{1}{2}+i\tau}$ is the Legendre function of first kind and order zero.

Related studies on Mehler-Fock transforms of generalized functions have been carried out in [3], [4], [5], [8], amongst others.

In this paper, we study the Mehler-Fock transform by using the method of adjoints. For this purpose, it will be required to introduce two Fréchet spaces V and W satisfying the condition that the linear operator $\mathcal{L} : W \rightarrow V$ given by

$$(1.2) \quad (\mathcal{L}\psi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x) \psi(\tau) d\tau, \quad x > 0,$$

be continuous.

Specifically, we denote by V the vector space consisting of all complex-valued functions ϕ which possess derivatives of all orders on I and such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\phi) = \sup_{x \in I} |(\sinh x) A_x^k \phi(x)| < \infty,$$

where A_x is the differential operator

$$A_x \equiv D_x^2 + (\coth x) D_x \equiv (\sinh x)^{-1} D_x (\sinh x) D_x,$$

and D_x denotes derivative with respect to the x -variable.

The topology of V is that generated by the family of seminorms $\{\gamma_k\}_{k \in \mathbb{N} \cup \{0\}}$.

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Moreover, W is the vector space of all complex-valued even functions ψ on $\mathbb{R}$ such that

$$\tau^{-1} \left(\frac{1}{4} + \tau^2 \right)^r \cosh(\pi\tau) \psi(\tau) \in L^1(\mathbb{R})$$

and

$$\rho_r(\psi) = \sup_{x \in I} \left| x^{\frac{3}{2}} e^x \left(\mathcal{F} \left[\tau^{-1} \left(\frac{1}{4} + \tau^2 \right)^r \cosh(\pi\tau) \psi(\tau) \right] \right) (x) \right| < \infty,$$

for all $r \in \mathbb{N} \cup \{0\}$, and $\mathcal{F}$ denotes Fourier transform with respect to the τ -variable. The space W is equipped with the topology generated by the family of seminorms $\{\rho_r\}_{r \in \mathbb{N} \cup \{0\}}$.

As usual V' and W' denote the dual spaces of V and W , respectively.

The Mehler-Fock transform of the generalized function $f \in V'$ is given by $\mathcal{L}'f$, where $\mathcal{L}'$ is the adjoint of $\mathcal{L}$.

Here, we establish the link between these two methods. It will be proved that both definitions agree for distributions of compact support on I .

We also establish a Paley-Wiener type theorem. It is proved that every smooth function F such that $|F(\tau)| \leq C \cdot (1 + \tau^{2r})$, for all $\tau > 0$, some nonnegative integer r and some constant C , is the Mehler-Fock transform of a generalized function $f \in V'$.

In this article we extend this analysis to the so-called Mehler-Fock transform of order $n \in \mathbb{N}$.

The corresponding results for the Kontorovich-Lebedev transform were carried out in [7].

2. THE OPERATOR $\mathcal{L}$ AND ITS ADJOINT

The next result is the key to analysing the Mehler-Fock transform of generalized functions via the method of adjoints.

Theorem 2.1. *The operator $\mathcal{L}$, given by (1.2), is a continuous linear mapping from W in V .*

Proof. First observe that for all $\tau > 0$,

$$A_x P_{-\frac{1}{2}+i\tau}(\cosh x) = - \left(\frac{1}{4} + \tau^2 \right) P_{-\frac{1}{2}+i\tau}(\cosh x).$$

So, for any $\psi \in W$ and all $k \in \mathbb{N} \cup \{0\}$, we have

$$(2.1) \quad A_x^k (\mathcal{L}\psi)(x) = (-1)^k \int_0^\infty \left(\frac{1}{4} + \tau^2 \right)^k P_{-\frac{1}{2}+i\tau}(\cosh x) \psi(\tau) d\tau.$$

Using the integral representation (cf. [1, (11), p. 156]), which is valid for all $x > 0$ and all $\tau > 0$,

$$(2.2) \quad P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\sqrt{2}}{\pi} \cosh(\pi\tau) \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{1}{2}} \cos(\tau\xi) d\xi$$

and integrating (2.2) by parts, one obtains

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\cosh(\pi\tau)}{\sqrt{2}\pi\tau} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sin(\tau\xi)) (\sinh \xi) d\xi.$$

Substituting this expression in (2.1) and reversing the order of integration, it follows that

$$(2.3) \quad A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2\pi}} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sinh \xi) \\ \times \left(\int_0^\infty \tau^{-1} \cosh(\pi\tau) \left(\frac{1}{4} + \tau^2 \right)^k \psi(\tau) \sin(\tau\xi) d\tau \right) d\xi.$$

Note that the integral into the brackets is the Fourier sine transform of an odd function of the τ -variable, and consequently it is the Fourier transform of this function.

Therefore,

$$(2.4) \quad A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2\pi}} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sinh \xi) \\ \times \left(\mathcal{F} \left[\tau^{-1} \cosh(\pi\tau) \left(\frac{1}{4} + \tau^2 \right)^k \psi(\tau) \right] \right) (\xi) d\xi.$$

Thus, for all $x > 0$ we have

$$|(\sinh x) A_x^k(\mathcal{L}\psi)(x)| = \left| \frac{\sinh x}{\sqrt{2\pi} (\cosh x)^{\frac{3}{2}}} \int_0^\infty \left(1 + \frac{\cosh \xi}{\cosh x} \right)^{-\frac{3}{2}} (\sinh \xi) \right. \\ \left. \times \left(\mathcal{F} \left[\tau^{-1} \cosh(\pi\tau) \left(\frac{1}{4} + \tau^2 \right)^k \psi(\tau) \right] \right) (\xi) d\xi \right|$$

which is less than or equal to

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\xi} \xi^{-\frac{3}{2}} (\sinh \xi) \left| e^{\xi} \xi^{\frac{3}{2}} \left(\mathcal{F} \left[\tau^{-1} \cosh(\pi\tau) \left(\frac{1}{4} + \tau^2 \right)^k \psi(\tau) \right] \right) (\xi) \right| d\xi \\ \leq M \rho_k(\psi),$$

where M is a suitable real constant. Thus, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\mathcal{L}\psi) \leq M \rho_k(\psi).$$

This concludes the continuity of $\mathcal{L}$. ■

Now, the adjoint of the operator $\mathcal{L}$, which is given by

$$\langle \mathcal{L}' f, \psi \rangle = \langle f, \mathcal{L}\psi \rangle, \quad f \in V', \quad \psi \in W,$$

is a continuous linear mapping from V' into W' when we provide V' and W' with their weak*-topologies. The Mehler-Fock transform of the generalized function f is the member of W' given by $\mathcal{L}' f$.

3. THE LINK BETWEEN BOTH METHODS

The next two lemmas allow us to establish the link between the two definitions of Mehler-Fock transforms of generalized functions.

For it, set

$$(3.1) \quad \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \phi(x) dx,$$

for any $\phi \in V$ and all $\tau > 0$.

Lemma 3.1. For all $\phi \in V$ the function ψ_ϕ given by (3.1) satisfies

- i) $\psi_\phi(\tau) = O(\tau^2)$, as $\tau \rightarrow 0$;
- ii) for all $p \in \mathbb{N}$, $\psi_\phi(\tau) = O(\tau^{-p})$, as $\tau \rightarrow \infty$.

Proof. i) Taking into account that $\tanh(\pi\tau) = O(\tau)$ for $\tau \rightarrow 0$, it follows that for τ small enough,

$$|\psi_\phi(\tau)| \leq C_1 \tau^2 \int_0^\infty |(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \phi(x)| dx \leq C_2 \tau^2,$$

for some suitable real constants C_1 and C_2 .

ii) If A'_x denotes the adjoint of A_x , namely, $A'_x = D_x(\sinh x)D_x(\sinh x)^{-1}$, one has

$$A'_x \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] = - \left(\frac{1}{4} + \tau^2 \right) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x),$$

and so, for all $k \in \mathbb{N} \cup \{0\}$,

(3.2)

$$(-1)^k \left(\frac{1}{4} + \tau^2 \right)^k \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (A'_x)^k \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] \phi(x) dx.$$

Note that the integral in the right-hand side of (3.2) is equal to

$$\int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x^k \phi(x) dx.$$

In fact, integrating by parts,

$$\begin{aligned} & \int_0^\infty A'_x \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] \phi(x) dx \\ &= (\sinh x) \left(D_x P_{-\frac{1}{2}+i\tau}(\cosh x) \right) \phi(x) \Big|_0^\infty \\ & \quad - \int_0^\infty (\sinh x) \left(D_x P_{-\frac{1}{2}+i\tau}(\cosh x) \right) (D_x \phi(x)) dx \\ &= (\sinh x) \left(D_x P_{-\frac{1}{2}+i\tau}(\cosh x) \right) \phi(x) \Big|_0^\infty \\ & \quad - (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) (D_x \phi(x)) \Big|_0^\infty \\ & \quad + \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x \phi(x) dx. \end{aligned}$$

Observe that the remainder terms of the above integration by parts tend to zero as $x \rightarrow 0$ and $x \rightarrow \infty$. Indeed, since $\phi \in V$, it follows that $(\sinh x)\phi(x)$ is bounded on I , and

$$D_x P_{-\frac{1}{2}+i\tau}(\cosh x) \longrightarrow 0, \text{ as } x \rightarrow 0 \text{ and } x \rightarrow \infty.$$

Hence,

$$(\sinh x) (D_x P_{-\frac{1}{2}+i\tau}(\cosh x)) \phi(x) \longrightarrow 0, \text{ as } x \rightarrow 0 \text{ and } x \rightarrow \infty.$$

Also, since for all $\phi \in V$, the function $(\sinh x)D_x \phi(x)$ is bounded on I , and taking into account that $P_{-\frac{1}{2}+i\tau}(\cosh x) = O(e^{-\frac{x}{2}})$ as $x \rightarrow \infty$, it follows that

$$(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) D_x \phi(x) \longrightarrow 0, \text{ as } x \rightarrow \infty.$$

On the other hand, for any $\phi \in V$ one has

$$(\sinh x)D_x\phi(x) = O(x), \text{ as } x \rightarrow 0.$$

Thus,

$$(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)D_x\phi(x) \longrightarrow 0, \text{ as } x \rightarrow 0.$$

This proves that

$$-\left(\frac{1}{4} + \tau^2\right)\psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)A_x\phi(x)dx.$$

Now, for all $k \in \mathbb{N}$, $k > 1$, one obtains

$$\begin{aligned} & \int_0^\infty (A'_x)^k \left[(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x) \right] \phi(x)dx \\ &= (-1)^{k-1} \left(\frac{1}{4} + \tau^2 \right)^{k-1} \int_0^\infty A'_x \left[(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x) \right] \phi(x)dx \\ &= (-1)^{k-1} \left(\frac{1}{4} + \tau^2 \right)^{k-1} \int_0^\infty (\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)A_x\phi(x)dx. \end{aligned}$$

Iterating this process one arrives at

$$(-1)^k \left(\frac{1}{4} + \tau^2 \right)^k \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)A_x^k\phi(x)dx.$$

Taking into account that $\phi \in V$ and the asymptotic behavior of $P_{-\frac{1}{2}+i\tau}(\cosh x)$ as $x \rightarrow 0$ and $x \rightarrow \infty$, one concludes that

$$\left(\frac{1}{4} + \tau^2 \right)^k |\psi_\phi(\tau)| \leq M\tau, \quad \forall \tau > 0,$$

for some suitable nonnegative constant M . Thus, the lemma holds. $\blacksquare$

Lemma 3.2. *For all $\psi \in W$, there exists $\phi \in V$ such that $\psi = \psi_\phi$, where ψ_ϕ is given by (3.1).*

Proof. Let $\psi \in W$ and set $\phi = \mathcal{L}\psi$. Note that, for $x > 0$, one has

$$\begin{aligned} (\mathcal{L}\psi_\phi)(x) &= \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi_\phi(\tau)d\tau \\ &= \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\tau \tanh(\pi\tau) \int_0^\infty (\sinh t)P_{-\frac{1}{2}+i\tau}(\cosh t)\phi(t)dt d\tau, \end{aligned}$$

and from the inversion formula of the Mehler-Fock transform, the above expression is equal to $\phi(x)$, which holds whenever $\phi(x)e^{\frac{x}{2}} \in L^1(0, \infty)$ (cf. [2]). Since $\phi \in V$, this condition is satisfied.

Thus,

$$\mathcal{L}\psi_\phi = \phi = \mathcal{L}\psi.$$

Now, we will establish that $\mathcal{L}$ is one to one. For it, assume that $\mathcal{L}\psi = 0$. Taking $k = 0$ in (2.4) one has for all $x > 0$,

$$(\mathcal{L}\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sinh \xi) (\mathcal{F} [\tau^{-1} \cosh(\pi\tau)\psi(\tau)]) (\xi) d\xi = 0.$$

The change of variables $t + 1 = \cosh \xi$ yields

$$0 = (\mathcal{L}\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty (\cosh x + 1 + t)^{-\frac{3}{2}} \Psi(t) dt, \quad \forall x > 0,$$

where

$$\Psi(t) = (\mathcal{F} [\tau^{-1} \cosh(\pi\tau) \psi(\tau)]) (\log(t + 1 + \sqrt{t^2 + 2t})).$$

Finally, since

$$\int_0^\infty (\cosh x + 1 + t)^{-\frac{3}{2}} \Psi(t) dt$$

is the generalized Stieltjes transform of the function Ψ , from its inversion formula (cf. [9, p. 180]) it follows that $\Psi \equiv 0$. Therefore,

$$(\mathcal{F} [\tau^{-1} \cosh(\pi\tau) \psi(\tau)]) (\log(t + 1 + \sqrt{t^2 + 2t})) = 0, \quad \forall t > 0,$$

whence $\psi \equiv 0$. ■

Theorem 3.1. *Let $f \in \mathcal{E}'(I)$. Then, for any $\psi \in W$, one has*

$$(3.3) \quad \langle \mathcal{L}' f, \psi \rangle = \int_0^\infty F(\tau) \psi(\tau) d\tau,$$

where the function $F(\tau)$ is given by (1.1).

Proof. First, observe that $\mathcal{D}(I) \subset V \subset \mathcal{E}(I)$ and the topology of V is stronger than the one induced on it by $\mathcal{E}(I)$. Thus, we may regard $\mathcal{E}'(I)$ as a vector subspace of V' .

By [6, Proposition 2, p. 97], the function $F(\tau)$ satisfies

i) For $\tau \rightarrow 0$,

$$(3.4) \quad F(\tau) = O(1),$$

ii) there exists $r \in \mathbb{N} \cup \{0\}$ such that for $\tau \rightarrow \infty$,

$$(3.5) \quad F(\tau) = O(\tau^{2r}).$$

Using Lemma 3.1 and Lemma 3.2 it follows that for any $\psi \in W$ one has

i) $\psi(\tau) = O(\tau^2)$, as $\tau \rightarrow 0$;

ii) for all $p \in \mathbb{N}$, $\psi(\tau) = O(\tau^{-p})$, as $\tau \rightarrow \infty$.

Thus, the integral of the right-hand side of (3.3) makes sense.

Clearly, if $f \in \mathcal{D}(I)$, the result of this theorem holds. In fact, assume that f has its support contained in the closed interval $[a, b] \subset I$; then, using Fubini's theorem, one has

$$\begin{aligned} \langle \mathcal{L}' f, \psi \rangle &= \langle f, \mathcal{L}\psi \rangle \\ &= \int_a^b f(x) (\mathcal{L}\psi)(x) dx = \int_a^b f(x) \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x) \psi(\tau) d\tau dx \\ &= \int_0^\infty \psi(\tau) \int_a^b f(x) P_{-\frac{1}{2}+i\tau}(\cosh x) dx d\tau = \int_0^\infty F(\tau) \psi(\tau) d\tau. \end{aligned}$$

So, the result holds for $f \in \mathcal{D}(I)$.

Now, for $f \in \mathcal{E}'(I)$ and using [11, Theorem 28.2(i), p. 301], there exists a sequence of functions $\{f_m\}_{m \in \mathbb{N}}$ belonging to $\mathcal{D}(I)$ which converges to f in $\mathcal{E}'(I)$,

and hence, in V' . Furthermore, from Theorem 2.1 it follows that $\mathcal{L}'$ is a continuous mapping from V' into W' . Thus $\{\mathcal{L}'f_m\}_{m \in \mathbb{N}}$ converges to $\mathcal{L}'f$ in W' as $m \rightarrow \infty$.

Since, for all $\psi \in W$,

$$\langle \mathcal{L}'f_m, \psi \rangle = \int_0^\infty F_m(\tau) \psi(\tau) d\tau$$

where

$$F_m(\tau) = \langle f_m(x), P_{-\frac{1}{2}+i\tau}(\cosh x) \rangle,$$

it follows that

$$(3.6) \quad \langle \mathcal{L}'f, \psi \rangle = \lim_{m \rightarrow \infty} \int_0^\infty F_m(\tau) \psi(\tau) d\tau.$$

Clearly, from the asymptotic growth of F_m and ψ , the limit in (3.6) goes into the integral. Also, noting that

$$\begin{aligned} \lim_{m \rightarrow \infty} F_m(\tau) &= \lim_{m \rightarrow \infty} \langle f_m(x), P_{-\frac{1}{2}+i\tau}(\cosh x) \rangle \\ &= \langle f(x), P_{-\frac{1}{2}+i\tau}(\cosh x) \rangle = F(\tau), \end{aligned}$$

the result holds. ■

4. A PALEY-WIENER TYPE THEOREM

The following theorem proves that any smooth function which satisfies conditions of type (3.4) and (3.5) is the Mehler-Fock transformation of an element of V' .

Theorem 4.1. *Let $F(\tau)$ be a smooth function on I such that there exist a nonnegative integer r and a constant C with $|F(\tau)| \leq C \cdot (1 + \tau^{2r})$ for all $\tau > 0$. Then there exists a $f \in V'$ such that*

$$\langle \mathcal{L}'f, \psi \rangle = \int_0^{+\infty} F(\tau) \psi(\tau) d\tau, \quad \forall \psi \in W.$$

Moreover, for all $\phi \in V$,

$$\langle f, \phi \rangle = \lim_{T \rightarrow \infty} \left\langle \int_0^T \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh \tau) F(\tau) d\tau, \phi(x) \right\rangle.$$

Proof. Let $G(\tau)$ be a smooth function on $(0, \infty)$ such that

$$(4.1) \quad |G(\tau)| \leq M_1 \cdot (1 + \tau^{-\alpha}),$$

where $\alpha > 2$ and M_1 is a real constant. Let us define, for any $x > 0$,

$$(4.2) \quad g(x) = \int_0^\infty \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau.$$

Note that for $\tau \rightarrow 0$,

$$\tau \tanh(\pi\tau) = O(\tau^2),$$

and for $\tau \rightarrow \infty$,

$$\tau \tanh(\pi\tau) = O(\tau).$$

On the other hand,

$$\left| P_{-\frac{1}{2}+i\tau}(\cosh x) \right| \leq \left| P_{-\frac{1}{2}}(\cosh x) \right|, \quad \forall x > 0, \quad \forall \tau > 0.$$

So, (4.2) exists and it is a locally integrable function on I . Moreover,

$$\begin{aligned} \int_0^\infty |(\sinh x)^{-1} g(x)| dx &= \int_0^\infty \left| \int_0^\infty \tau \tanh(\pi\tau) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau \right| dx \\ &\leq \int_0^\infty \left| P_{-\frac{1}{2}}(\cosh x) \right| \int_0^\infty \tau |\tanh(\pi\tau)| |G(\tau)| d\tau dx \\ &\leq M_2 \cdot \int_0^\infty \left| P_{-\frac{1}{2}}(\cosh x) \right| dx < \infty, \end{aligned}$$

where M_2 is a real constant.

Therefore, the function g yields to a regular member in V' , denoted by T_g , which is given by

$$\langle T_g, \phi \rangle = \int_0^\infty g(x) \phi(x) dx, \quad \forall \phi \in V.$$

Also for any $\psi \in W$ denoting $\phi = \mathcal{L}\psi$, and using Lemma 3.2, one obtains

$$\begin{aligned} \langle \mathcal{L}' T_g, \psi \rangle &= \langle T_g, \mathcal{L}\psi \rangle = \int_0^\infty g(x) \phi(x) dx \\ &= \int_0^\infty \phi(x) \int_0^\infty \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau dx, \end{aligned}$$

which, by Fubini's theorem, is equal to

$$\int_0^\infty G(\tau) \int_0^\infty \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \phi(x) dx d\tau = \int_0^\infty G(\tau) \psi(\tau) d\tau.$$

Thus, for all $\psi \in W$,

$$(4.3) \quad \langle \mathcal{L}' T_g, \psi \rangle = \int_0^\infty G(\tau) \psi(\tau) d\tau.$$

Now, for all $\tau > 0$, set

$$G(\tau) = \frac{F(\tau)}{\left(\frac{1}{4} + \tau^2\right)^{r+p}}.$$

Clearly, for $p > \alpha/2$, this function G satisfies condition (4.1). So, there exists a function g given by (4.2) such that the regular member T_g in V' satisfies (4.3). Set $f = (-A'_x)^{r+p} T_g$. Note that, for all $\phi \in V$,

$$\langle f, \phi \rangle = \langle (-A'_x)^{r+p} T_g, \phi \rangle = \int_0^\infty g(x) (-A_x)^{r+p} \phi(x) dx.$$

Thus, for all $\psi \in W$, one obtains

$$\begin{aligned} \langle \mathcal{L}' f, \psi \rangle &= \langle T_g, (-A_x)^{r+p} (\mathcal{L}\psi) \rangle = \left\langle T_g, \mathcal{L} \left[\left(\frac{1}{4} + \tau^2 \right)^{r+p} \psi(\tau) \right] \right\rangle \\ &= \int_0^\infty G(\tau) \left(\frac{1}{4} + \tau^2 \right)^{r+p} \psi(\tau) d\tau = \int_0^\infty F(\tau) \psi(\tau) d\tau. \end{aligned}$$

On the other hand, for any $\phi \in V$,

$$\langle f, \phi \rangle = \left\langle \int_0^\infty \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau, (-A_x)^{r+p} \phi(x) \right\rangle$$

$$\begin{aligned}
&= \lim_{T \rightarrow \infty} \left\langle \int_0^T \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau, (-A_x)^{r+p} \phi(x) \right\rangle \\
&= \lim_{T \rightarrow \infty} \left\langle \int_0^T \tau \tanh(\pi\tau) (-A'_x)^{r+p} \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \right] G(\tau) d\tau, \phi(x) \right\rangle \\
&= \lim_{T \rightarrow \infty} \left\langle \int_0^T \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \left(\frac{1}{4} + \tau^2 \right)^{r+p} G(\tau) d\tau, \phi(x) \right\rangle \\
&= \lim_{T \rightarrow \infty} \left\langle \int_0^T \tau \tanh(\pi\tau) (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) F(\tau) d\tau, \phi(x) \right\rangle.
\end{aligned}$$

■

5. MEHLER-FOCK TRANSFORMS OF ORDER n

Starting from the integral representation [1, (11), p. 156], which is valid for all $n \in \mathbb{N} \cup \{0\}$, all $\tau > 0$ and all $x > 0$,

$$\begin{aligned}
&P_{-\frac{1}{2}+i\tau}^n(\cosh x) \\
&= \sqrt{\frac{2}{\pi}} \frac{\Gamma(n + \frac{1}{2})(\sinh x)^n}{\Gamma(n + \frac{1}{2} + i\tau)\Gamma(n + \frac{1}{2} - i\tau)} \int_0^\infty (\cosh x + \cosh \xi)^{-n-\frac{1}{2}} \cos(\tau\xi) d\xi,
\end{aligned}$$

the results of the previous sections are readily extended to the so-called Mehler-Fock transform of order $n \in \mathbb{N}$, the kernel of which is given by $P_{-\frac{1}{2}+i\tau}^n(\cosh x)$, the associated Legendre function of first kind and order n .

Specifically, one introduces the corresponding spaces V and W ; namely, V is the vector space consisting of all complex-valued functions ϕ which possess derivatives of all orders on I and such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\phi) = \sup_{x \in I} |(\sinh x) A_x^k \phi(x)| < \infty,$$

where A_x is the differential operator

$$A_x \equiv (\sinh x)^{-n-1} D_x (\sinh x)^{2n+1} D_x (\sinh x)^{-n},$$

and W is the vector space of all complex-valued even functions ψ on $\mathbb{R}$ such that

$$\frac{\left(\left(n + \frac{1}{2} \right)^2 + \tau^2 \right)^r \psi(\tau)}{\tau \Gamma \left(n + \frac{1}{2} + i\tau \right) \Gamma \left(n + \frac{1}{2} - i\tau \right)} \in L^1(\mathbb{R})$$

and

$$\rho_r(\psi) = \sup_{x \in I} \left| x^{\frac{3}{2}} e^x \left(\mathcal{F} \left[\frac{\left(\left(n + \frac{1}{2} \right)^2 + \tau^2 \right)^r \psi(\tau)}{\tau \Gamma \left(n + \frac{1}{2} + i\tau \right) \Gamma \left(n + \frac{1}{2} - i\tau \right)} \right] \right) (x) \right| < \infty,$$

for all $r \in \mathbb{N} \cup \{0\}$, and $\mathcal{F}$ denotes Fourier transform with respect to the τ -variable.

The operator $\mathcal{L} : W \rightarrow V$, given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}^n(\cosh x) \psi(\tau) d\tau,$$

becomes a continuous linear mapping from W in V .

Also, for any $\phi \in V$ and all $\tau > 0$ set

$$\begin{aligned}\psi_\phi(\tau) &= \frac{1}{\pi} \tau \sinh(\pi\tau) \Gamma\left(n + \frac{1}{2} + i\tau\right) \Gamma\left(n + \frac{1}{2} - i\tau\right) \\ &\quad \times \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}^n(\cosh x) \phi(x) dx.\end{aligned}$$

It is obtained that

i) $\psi_\phi(\tau) = O(\tau^{2n+2})$, as $\tau \rightarrow 0$;

ii) for all $p \in \mathbb{N}$, $\psi_\phi(\tau) = O(\tau^{-p})$, as $\tau \rightarrow \infty$.

Also, for any $\psi \in W$ there exists $\phi \in V$ such that $\psi = \psi_\phi$.

Now, for any $f \in \mathcal{E}'(I)$, the function

$$F(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}^n(\cosh x) \rangle, \quad \tau > 0,$$

satisfies

i) $F(\tau) = O(1)$, as $\tau \rightarrow 0$;

ii) there exists $r \in \mathbb{N} \cup \{0\}$ such that $F(\tau) = O(\tau^{2r})$, as $\tau \rightarrow \infty$.

Thus the corresponding result of Theorem 3.1 holds in this setting; namely, for all $f \in \mathcal{E}'(I)$,

$$\langle \mathcal{L}' f, \psi \rangle = \int_0^\infty F(\tau) \psi(\tau) d\tau.$$

Also the corresponding Paley-Wiener type theorem establishes that

Theorem 5.1. *Let $F(\tau)$ be a smooth function on I such that there exist a nonnegative integer r and a constant C with $|F(\tau)| \leq C \cdot (1 + \tau^{2r})$ for all $\tau > 0$. Then there exists an $f \in V'$ such that*

$$\langle \mathcal{L}' f, \psi \rangle = \int_0^{+\infty} F(\tau) \psi(\tau) d\tau, \quad \forall \psi \in W.$$

Moreover, for all $\phi \in V$,

$$\langle f, \phi \rangle = \lim_{T \rightarrow \infty} \left\langle \int_0^T S(\tau) G(\tau, x) F(\tau) d\tau, \phi(x) \right\rangle,$$

where

$$S(\tau) = \frac{1}{\pi} \tau \sinh(\pi\tau) \Gamma\left(n + \frac{1}{2} + i\tau\right) \Gamma\left(n + \frac{1}{2} - i\tau\right)$$

and

$$G(\tau, x) = (\sinh x) \cdot P_{-\frac{1}{2}+i\tau}^{-n}(\cosh x).$$

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