STANDARD SYSTEMS FOR SEMIFINITE O^* -ALGEBRAS

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ABSTRACT. We shall continue the study of standard systems which make it possible to develop the Tomita-Takesaki theory in O^* -algebras. The main purpose of this paper is to give the necessary and sufficient conditions for which a standard system $(\mathcal{M}, \lambda, \lambda')$ of an O^* -algebra $\mathcal{M}$, a generalized vector λ and the commutant λ' is unitarily equivalent to a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$ constructed by a standard tracial generalized vector μ for an O^* -algebra $\mathcal{N}$ and a non-singular positive self-adjoint operator K' affiliated with the commutant $\mathcal{N}'_w$ of $\mathcal{N}$.

1. INTRODUCTION

In order to develop the Tomita-Takesaki theory in O^* -algebras, we have defined and studied the notion of standard system $(\mathcal{M}, \lambda, \lambda')$ of an O^* -algebra $\mathcal{M}$, a generalized vector λ and the commutant λ' in [1, 6, 7]. Here we shall continue this study and study the structure of standard systems for semifinite O^* -algebras. We first treat tracial generalized vectors μ for an O^* -algebra $\mathcal{N}$ and give the necessary and sufficient conditions for which μ is standard. We next construct a generalized vector $K'\mu$ by a pair (K', μ) of a standard tracial generalized vector μ for $\mathcal{N}$ and a non-singular positive self-adjoint operator K' affiliated with the commutant $\mathcal{N}'_w$ of $\mathcal{N}$, and investigate when $(\mathcal{N}, K'\mu, (K'\mu)')$ is a standard system. Further, we consider this converse. And we give the necessary and sufficient conditions for which a standard system $(\mathcal{M}, \lambda, \lambda')$ is unitarily equivalent to such a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$.

2. STANDARD TRACIAL GENERALIZED VECTORS

We begin with the definitions and the basic properties of standard generalized vectors for O^* -algebras [7]. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{D}(X^*)$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$ -algebra equipped with the usual operators $X+Y$, αX and the involution $X^\dagger \equiv X^*|_{\mathcal{D}}$. A $*$ -subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$. An O^* -algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$ is said to be *closed* (resp. *self-adjoint*) if

$$\mathcal{D} = \bigcap_{X \in \mathcal{M}} \mathcal{D}(\overline{X}) \quad \left(\text{resp. } \mathcal{D} = \bigcap_{X \in \mathcal{M}} \mathcal{D}(X^*) \right),$$

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and $\mathcal{M}$ is said to be *integrable* if $X^{\dagger*} = \overline{X}$ for each $X \in \mathcal{M}$. It is clear that if $\mathcal{M}$ is integrable, then it is self-adjoint, and if $\mathcal{M}$ is self-adjoint, then it is closed. We define the *weak commutant* $\mathcal{M}'_w$ of an O^* -algebra $\mathcal{M}$ by

$$\mathcal{M}'_w = \{C \in \mathcal{B}(\mathcal{H}) ; (CX\xi|\eta) = (C\xi|X^\dagger\eta) \text{ for each } X \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\},$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. It is known that if $\mathcal{M}$ is self-adjoint, then $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$; and if $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$, then $\mathcal{M}'_w$ is a von Neumann algebra. An O^* -algebra $\mathcal{M}$ is said to be *semifinite* if $(\mathcal{M}'_w)'$ is a semifinite von Neumann algebra. For the general theory of O^* -algebras we refer to [10, 14]. We introduce the notion of generalized vectors for O^* -algebras. Let $\mathcal{M}$ be a closed O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$. A *generalized vector* λ for $\mathcal{M}$ is a linear map of the left ideal $\mathcal{D}(\lambda)$ of $\mathcal{M}$ into $\mathcal{D}$ satisfying $\lambda(XY) = X\lambda(Y)$ for each $X \in \mathcal{M}$ and $Y \in \mathcal{D}(\lambda)$; and a generalized vector λ is said to be *tracial* if $(\lambda(X)|\lambda(Y)) = (\lambda(Y^\dagger)|\lambda(X^\dagger))$ for each $X, Y \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$. Let λ be a generalized vector for $\mathcal{M}$. Suppose

(S)₁ $\lambda((\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))^2)$ is total in $\mathcal{H}$. Then we define two commutants λ' and λ^C of λ as follows:

$$\begin{cases} \mathcal{D}(\lambda') = \{C \in \mathcal{M}'_w ; \exists \xi_C \in \bigcap_{X \in \mathcal{D}(X)} \mathcal{D}(\overline{X}) \text{ s.t. } C\lambda(X) = \overline{X}\xi_C, \forall X \in \mathcal{D}(\lambda)\}, \\ \lambda'(C) = \xi_C, \quad C \in \mathcal{D}(\lambda'), \end{cases}$$

$$\begin{cases} \mathcal{D}(\lambda^C) = \{C \in \mathcal{M}'_w ; \exists \xi_C \in \mathcal{D} \text{ s.t. } C\lambda(X) = X\xi_C, \forall X \in \mathcal{D}(\lambda)\}, \\ \lambda^C(C) = \xi_C, \quad C \in \mathcal{D}(\lambda^C), \end{cases}$$

and then λ' and λ^C are generalized vectors for the von Neumann algebra $\mathcal{M}'_w$ such that $\lambda^C \subset \lambda'$, that is, $\mathcal{D}(\lambda^C) \subset \mathcal{D}(\lambda')$ and $\lambda^C(C) = \lambda'(C)$, $\forall C \in \mathcal{D}(\lambda^C)$.

Definition 2.1. Let λ be a generalized vector for $\mathcal{M}$. Suppose λ satisfies the above condition (S)₁ and the following condition (S)₂' (resp. (S)₂^C). Then $(\mathcal{M}, \lambda, \lambda')$ (resp. $(\mathcal{M}, \lambda, \lambda^C)$) is said to be a cyclic and separating system:

(S)₂' $\lambda'((\mathcal{D}(\lambda')^* \cap \mathcal{D}(\lambda'))^2)$ is total in $\mathcal{H}$.

(S)₂^C $\lambda^C((\mathcal{D}(\lambda^C)^* \cap \mathcal{D}(\lambda^C))^2)$ is total in $\mathcal{H}$.

When $(\mathcal{M}, \lambda, \lambda^C)$ is a cyclic and separating system, λ is said to be a cyclic and separating generalized vector.

Suppose $(\mathcal{M}, \lambda, \lambda')$ is a cyclic and separating system and put

$$\begin{cases} \mathcal{D}(\lambda'') = \{A \in (\mathcal{M}'_w)' ; \exists \xi_A \in \mathcal{H} \text{ s.t. } A\lambda'(C) = C\xi_A, \forall C \in \mathcal{D}(\lambda')\}, \\ \lambda''(A) = \xi_A, \quad A \in \mathcal{D}(\lambda''). \end{cases}$$

Then λ'' is a cyclic and separating generalized vector for the von Neumann algebra $(\mathcal{M}'_w)'$, so that the maps $\lambda(X) \rightarrow \lambda(X^\dagger)$, $X \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$ and $\lambda''(A) \rightarrow \lambda''(A^*)$, $A \in \mathcal{A}(\lambda'')^* \cap \mathcal{D}(\lambda'')$ are closable in $\mathcal{H}$ and their closures are denoted by S_λ and $S_{\lambda''}$, respectively. Let $S_\lambda = J_\lambda \Delta_\lambda^{\frac{1}{2}}$ and $S_{\lambda''} = J_{\lambda''} \Delta_{\lambda''}^{\frac{1}{2}}$ be the polar decompositions of S_λ and $S_{\lambda''}$, respectively. Then it is shown that $S_\lambda \subset S_{\lambda''}$, and $J_{\lambda''}(\mathcal{M}'_w)' J_{\lambda''} = \mathcal{M}'_w$ and $\Delta_{\lambda''}^{it}(\mathcal{M}'_w)' \Delta_{\lambda''}^{-it} = (\mathcal{M}'_w)'$ for all $t \in \mathbb{R}$ by the Tomita fundamental theorem. But, we don't know how the unitary group $\{\Delta_{\lambda''}^{it}\}_{t \in \mathbb{R}}$ acts on the O^* -algebra $\mathcal{M}$, and so we define a system which has the best condition:

Definition 2.2. A cyclic and separating system $(\mathcal{M}, \lambda, \lambda')$ is said to be standard if the following conditions (S)₃' and (S)₄' hold:

- (S)'₃ $\Delta_{\lambda''}^{it} \mathcal{D} \subset \mathcal{D}$ and $\Delta_{\lambda''}^{it} \mathcal{M} \Delta_{\lambda''}^{-it} = \mathcal{M}$, $\forall t \in \mathbb{R}$.
 (S)'₄ $\Delta_{\lambda''}^{it} (\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)) \Delta_{\lambda''}^{-it} = \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$, $\forall t \in \mathbb{R}$.

By ([7], Theorem 5.5) we have the following

Theorem 2.3. *Suppose $(\mathcal{M}, \lambda, \lambda')$ is a standard system. Then the following statements hold:*

- (1) $S_\lambda = S_{\lambda''}$, and so $J_\lambda = J_{\lambda''}$ and $\Delta_\lambda = \Delta_{\lambda''}$.
 (2) $\{\sigma_t^\lambda\}_{t \in \mathbb{R}}$ is a one-parameter group of $*$ -automorphisms of $\mathcal{M}$, where $\sigma_t^\lambda(X) \equiv \Delta_\lambda^{it} X \Delta_\lambda^{-it}$, $X \in \mathcal{M}$, $t \in \mathbb{R}$.
 (3) λ satisfies the KMS-condition with respect to $\{\sigma_t^\lambda\}_{t \in \mathbb{R}}$; that is, for any $X, Y \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$ there exists an element $f_{X,Y}$ of $A(0,1)$ such that $f_{X,Y}(t) = (\lambda(\sigma_t^\lambda(X)) | \lambda(Y))$ and $f_{X,Y}(t+i) = (\lambda(Y^\dagger) | \lambda(\sigma_t^\lambda(X^\dagger)))$ for all $t \in \mathbb{R}$, where $A(0,1)$ is the set of all complex-valued functions, bounded and continuous on $0 \leq \text{Im} z \leq 1$ and analytic in the interior.

Suppose $(\mathcal{M}, \lambda, \lambda^C)$ is a cyclic and separating system and put

$$\begin{cases} \mathcal{D}(\lambda^{CC}) = \{A \in (\mathcal{M}'_w)'; \exists \xi_A \in \mathcal{H} \text{ s.t. } A\lambda^C(C) = C\xi_A, \forall C \in \mathcal{D}(\lambda^C)\}, \\ \lambda^{CC}(A) = \xi_A, A \in \mathcal{D}(\lambda^{CC}). \end{cases}$$

Then λ^{CC} is a cyclic and separating generalized vector for the von Neumann algebra $(\mathcal{M}'_w)'$, and so the closure of the closable involution $\lambda^{CC}(A) \rightarrow \lambda^{CC}(A^*)$, $A \in \mathcal{D}(\lambda^{CC})^* \cap \mathcal{D}(\lambda^{CC})$, is denoted by $S_{\lambda^{CC}}$ and let $S_{\lambda^{CC}} = J_{\lambda^{CC}} \Delta_{\lambda^{CC}}^{\frac{1}{2}}$ be the polar decomposition of $S_{\lambda^{CC}}$. Since $\lambda^C \subset \lambda'$, it follows that $(\mathcal{M}, \lambda, \lambda')$ is a cyclic and separating system and $S_\lambda \subset S_{\lambda''} \subset S_{\lambda^{CC}}$.

Definition 2.4. A cyclic and separating system $(\mathcal{M}, \lambda, \lambda^C)$ is said to be standard if the following conditions (S)₃^C and (S)₄^C hold:

- (S)₃^C $\Delta_{\lambda^{CC}}^{it} \mathcal{D} \subset \mathcal{D}$ and $\Delta_{\lambda^{CC}}^{it} \mathcal{M} \Delta_{\lambda^{CC}}^{-it} = \mathcal{M}$, $\forall t \in \mathbb{R}$.
 (S)₄^C $\Delta_{\lambda^{CC}}^{it} (\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)) \Delta_{\lambda^{CC}}^{-it} = \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$, $\forall t \in \mathbb{R}$.

By ([7], Theorem 5.6) we have the following

Theorem 2.5. *Suppose λ is a standard generalized vector for $\mathcal{M}$. Then $(\mathcal{M}, \lambda, \lambda')$ is a standard system satisfying $S_\lambda = S_{\lambda''} = S_{\lambda^{CC}}$.*

Any element ξ of $\mathcal{D}$ is regarded as generalized vector for $\mathcal{M}$ by $\mathcal{D}(\lambda_\xi) = \mathcal{M}$ and $\lambda_\xi(X) = X\xi$, $X \in \mathcal{M}$. It is easily shown that if ξ is *cyclic*, that is, $\mathcal{M}\xi$ is dense in $\mathcal{H}$ (iff λ_ξ is cyclic), then $\mathcal{D}(\lambda_\xi^C) = \mathcal{D}(\lambda'_\xi) = \mathcal{M}'_w$ and $\lambda_\xi^C(C) = \lambda'_\xi(C) = C\xi$, $\forall C \in \mathcal{M}'_w$. When both $\mathcal{M}\xi$ and $\mathcal{M}'_w\xi$ are dense in $\mathcal{H}$, we simply denote S_{λ_ξ} , J_{λ_ξ} and Δ_{λ_ξ} (resp. $S_{\lambda_\xi''}$, $J_{\lambda_\xi''}$ and $\Delta_{\lambda_\xi''}$) by S_ξ , J_ξ and Δ_ξ (resp. S_ξ'' , J_ξ'' and Δ_ξ''), respectively. If λ_ξ is standard, then ξ is said to be a *standard vector* for $\mathcal{M}$. If λ_ξ is tracial, then ξ is said to be a *tracial vector* for $\mathcal{M}$.

For the standardness of tracial generalized vectors we have the following

Proposition 2.6. *Let $\mathcal{M}$ be a closed O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$. Suppose μ is a tracial generalized vector for $\mathcal{M}$ such that $\mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{H}$. Then the following statements are equivalent:*

- (i) μ is standard; that is, $(\mathcal{M}, \mu, \mu^C)$ is a standard system.
 (ii) $\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$.

- (iii) $(\mathcal{M}, \mu, \mu')$ is a standard system.
- (iv) $\mu'((\mathcal{D}(\mu')^* \cap \mathcal{D}(\mu'))^2)$ is total in $\mathcal{H}$.
- (v) $J_\mu(\mathcal{M}'_w)'J_\mu = \mathcal{M}'_w$, where J_μ is the unitary involution on $\mathcal{H}$ defined by $J_\mu\mu(Y) = \mu(Y^\dagger)$ for each $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$.

If this is true, then $J_\mu = J_{\mu^{CC}} = J_{\mu''}$ and $\Delta_\mu = \Delta_{\mu^{CC}} = \Delta_{\mu''} = I$, and further $Y^{\dagger*} = \overline{Y}$ for each $Y \in \mathcal{D}(\mu)$.

Proof. We show the equivalence of (i) $\sim$ (v) by (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii). We show the implication (v) $\Rightarrow$ (ii). The other implications are trivial. Let $X \in \mathcal{M}$, and let $\overline{X} = U_X|\overline{X}|$, the polar decomposition of $\overline{X}$, and $|\overline{X}| = \int_0^\infty t dE_X(t)$, the spectral resolution of $|\overline{X}|$. We put $E_X(n) = \int_0^n dE_X(t)$ and $X_n = \overline{X}E_X(n)$, $n \in \mathbb{N}$. Then we have $U_X, E_X(n), X_n \in (\mathcal{M}'_w)'$ for each $n \in \mathbb{N}$. Take arbitrary $Y \in \mathcal{D}(\mu)$ and $n \in \mathbb{N}$. Then, it is not difficult to show

$$(2.1) \quad \begin{cases} J_\mu Y_n J_\mu \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C), \mu^C(J_\mu Y_n J_\mu) = J_\mu U_Y E_Y(n) U_Y^* J_\mu \mu(Y^\dagger), \\ \mu^C(J_\mu Y_n^* J_\mu) = J_\mu E_Y(n) J_\mu \mu(Y), \end{cases}$$

$$(2.2) \quad \lim_{n \rightarrow \infty} \mu^C(J_\mu Y_n J_\mu) = \mu(Y^\dagger).$$

Hence it follows that $\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$ since $\mu((\mathcal{D}(\mu)^* \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{H}$. Thus the statements (i) $\sim$ (v) are equivalent and $\Delta_{\mu''} = \Delta_{\mu^{CC}} = I$ and $J_{\mu''} = J_{\mu^{CC}} = J_\mu$.

We finally show that $Y^{\dagger*} = \overline{Y}$ for each $Y \in \mathcal{D}(\mu)$. By (2.1) and (2.2) we can show by simple calculations

$$(2.3) \quad \begin{aligned} \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C) &= \{J_\mu A^* J_\mu ; A \in \mathfrak{A}\}, \\ \mu^C(J_\mu A^* J_\mu) &= \mu^{CC}(A), \quad \mu^C(J_\mu A J_\mu) = \mu^{CC}(A^*), \quad A \in \mathfrak{A}, \end{aligned}$$

where $\mathfrak{A} \equiv \{A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}) ; \mu^{CC}(A), \mu^{CC}(A^*) \in \mathcal{D}\}$. Take an arbitrary $Y \in \mathcal{D}(\mu)$. By (2.3) and (2.2) we have

$$\pi_0(\mu(Y))\mu^{CC}(A) \equiv J_\mu A^* J_\mu \mu(Y) = Y\mu^{CC}(A),$$

$$\begin{aligned} \pi_0(J_\mu \mu(Y))\mu^{CC}(A) &\equiv J_\mu A^* \mu(Y) = \lim_{n \rightarrow \infty} J_\mu A^* \mu^C(J_\mu Y_n^* J_\mu) \\ &= \lim_{n \rightarrow \infty} Y_n^* \mu^{CC}(A) = Y^\dagger \mu^{CC}(A) \end{aligned}$$

for each $A \in \mathfrak{A}$, and further since $\mathfrak{A}$ is a Hilbert algebra in $\mathcal{H}$ by (2.3), it follows that $\overline{\pi_0(\mu(Y))} \subset \overline{Y}$ and $\overline{\pi_0(J_\mu \mu(Y))} \subset \overline{Y^\dagger}$. Further, it follows from the theory of Hilbert algebra [9] that $\pi_0(\xi)^* = \pi_0(J_\mu \xi)$, $\forall \xi \in \mathcal{H}$, that $\overline{\pi_0(\mu(Y))} \subset \overline{Y} \subset Y^{\dagger*} \subset \pi_0(J_\mu \mu(Y))^* = \pi_0(\mu(Y))$. Hence we have $Y^{\dagger*} = \overline{Y} = \pi_0(\mu(Y))$ for each $Y \in \mathcal{D}(\mu)$. This completes the proof. $\square$

By Proposition 2.6 we have the following

Corollary 2.7. *Let $\mathcal{M}$ be a closed O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$ and $\xi_0 \in \mathcal{D}$. Suppose ξ_0 is a cyclic tracial vector for $\mathcal{M}$. Then the following statements*

are equivalent:

- (i) ξ_0 is standard.
- (ii) $\mathcal{M}'_{\mathbf{w}}\xi_0$ is dense in $\mathcal{H}$.
- (iii) $J_{\xi_0}(\mathcal{M}'_{\mathbf{w}})'J_{\xi_0} = \mathcal{M}'_{\mathbf{w}}$.

If this is true, then $J_{\xi_0} = J''_{\xi_0}$, $\Delta_{\xi_0} = \Delta''_{\xi_0} = I$ and $\mathcal{M}$ is an integrable O^* -algebra on $\mathcal{D}$. Further, $\overline{\mathcal{M}} \equiv \{\overline{X} ; X \in \mathcal{M}\}$ is a $*$ -subalgebra of the $*$ -algebra $L^\omega(\omega_{\xi_0}) \equiv \bigcap_{1 \leq p < \infty} L^p(\omega_{\xi_0})$ equipped with the strong sum, strong scalar multiplication, strong product and adjoint, where $L^p(\omega_{\xi_0})$ is the Segal L^p -space with respect to the vector trace ω_{ξ_0} on $(\mathcal{M}'_{\mathbf{w}})'$ (refer to [4]).

3. STANDARD SYSTEMS FOR SEMIFINITE O^* -ALGEBRAS

In this section we treat a standard system $(\mathcal{M}, K'\mu, (K'\mu)')$ constructed by a standard tracial generalized vector μ and a non-singular positive self-adjoint operator K' , and consider when a standard system $(\mathcal{M}, \lambda, \lambda')$ is unitarily equivalent to such a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$. Let $\mathcal{M}$ be a closed O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\mathbf{w}}\mathcal{D} \subset \mathcal{D}$, μ a standard tracial generalized vector for $\mathcal{M}$ and K' a non-singular positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\mathbf{w}}$ whose domain $\mathcal{D}(K')$ contains $\mu(\mathcal{D}(\mu))$. Let $K' = \int_0^\infty t dE'(t)$ and $K \equiv J_\mu K' J_\mu = \int_0^\infty t dE(t)$ be the spectral resolutions of K' and K , respectively and let $E'(n) = \int_0^n dE'(t)$ and $E(n) = \int_0^n dE(t)$ for $n \in \mathbb{N}$. Here we put

$$\begin{cases} \mathcal{D}(K'\mu) = \mathcal{D}(\mu), \\ (K'\mu)(X) = K'\mu(X), \quad X \in \mathcal{D}(\mu). \end{cases}$$

Then it is easily shown that $K'\mu$ is a generalized vector for $\mathcal{M}$. For the standardness of the system $(\mathcal{M}, K'\mu, (K'\mu)')$ we have the following

Proposition 3.1. *Let $\mathcal{M}$ be a closed O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\mathbf{w}}\mathcal{D} \subset \mathcal{D}$ and μ a standard tracial generalized vector for $\mathcal{M}$. Suppose K' is a non-singular positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\mathbf{w}}$ such that $\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$, $K'\mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{H}$ and $K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$. Then $K'\mu$ is a generalized vector for $\mathcal{M}$ satisfying the following conditions:*

- (i) $(K'\mu)'((\mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2)$ is total in $\mathcal{H}$.
- (ii) $S_{K'\mu} = S_{(K'\mu)''} = J_\mu K \cdot K'^{-1}$.

Further, $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system if and only if $K^{it}\mathcal{D} \subset \mathcal{D}$ and $K^{it}YK^{-it} \upharpoonright \mathcal{D} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$.

Proof. Since μ is standard, there exists a net $\{A_\alpha\}$ in $\mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC})$ which converges strongly* to I . Let $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$ and let $Y_n = \overline{Y}E_Y(n)$, $n \in \mathbb{N}$. Then it follows from ([7], Lemma 5.2) that $\{Y_n\} \subset \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC})$, $Y_n \rightarrow Y$ strongly*, $\lim_{n \rightarrow \infty} \mu(Y_n) = \mu(Y)$ and $\lim_{n \rightarrow \infty} \mu^{CC}(Y_n^*) = \mu(Y^\dagger)$. Take arbitrary $C \in \mathcal{D}(\mu^C)$, $Y \in$

$\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$ and $m, n \in \mathbb{N}$. Then we have

$$\begin{aligned}
 E'(n)CE'(m)(K'\mu)(Y) &= \lim_{k \rightarrow \infty} E'(n)CE'(m)K'\mu^{CC}(Y_k) \\
 &= \lim_{k \rightarrow \infty} E'(n)CJ_\mu KE(m)\mu^{CC}(Y_k^*) \\
 &= \lim_{k \rightarrow \infty} \lim_{\alpha} E'(n)CJ_\mu A_\alpha KE(m)\mu^{CC}(Y_k^*) \\
 &= \lim_{k \rightarrow \infty} \lim_{\alpha} E'(n)CJ_\mu \mu^{CC}(A_\alpha KE(m)Y_k^*) \\
 &= \lim_{k \rightarrow \infty} \lim_{\alpha} E'(n)C\mu^{CC}(Y_k KE(m)A_\alpha^*) \\
 &= \lim_{k \rightarrow \infty} \lim_{\alpha} Y_k E'(n)KE(m)A_\alpha^* \mu^C(C) \\
 &= \lim_{k \rightarrow \infty} Y_k E'(n)KE(m)\mu^C(C),
 \end{aligned}$$

and so

$$\begin{aligned}
 (Y^\dagger \eta | E'(n)KE(m)\mu^C(C)) &= \lim_{k \rightarrow \infty} (Y_k^* \eta | E'(n)KE(m)\mu^C(C)) \\
 &= \lim_{k \rightarrow \infty} (\eta | Y_k E'(n)KE(m)\mu^C(C)) \\
 &= (\eta | E'(n)CE'(m)(K'\mu)(Y)),
 \end{aligned}$$

which implies $E'(n)KE(m)\mu^C(C) \in \mathcal{D}(Y^{\dagger*}) = \mathcal{D}(\overline{Y})$,

$$\overline{Y}E'(n)KE(m)\mu^C(C) = E'(n)CE'(m)(K'\mu)(Y).$$

Hence we have

$$\begin{aligned}
 E'(n)CE'(m) &\in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\
 (3.1) \quad (K'\mu)'(E'(n)CE'(m)) &= E'(n)KE(m)\mu^C(C), \\
 &\quad \forall C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C), \forall m, n \in \mathbb{N}.
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 E'(n)CE'(m)K'^{-1} &\in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\
 (3.2) \quad (K'\mu)'(E'(n)CE'(m)K'^{-1}) &= E'(n)E(m)\mu^C(C), \\
 (K'\mu)'(E'(m)K'^{-1}C^*E'(n)) &= E'(m)K'^{-1}E(n)K\mu^C(C) \\
 &\quad \forall C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C), m, n \in \mathbb{N}.
 \end{aligned}$$

By (3.1) and (3.2) we have

$$\begin{aligned}
 E'(n)C_1E'(l)C_2E'(m)K'^{-1} &\in (\mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2, \\
 \lim_{m, n \rightarrow \infty} (K'\mu)'(E'(n)C_1E'(l)C_2E'(m)K'^{-1}) \\
 &= \lim_{m, n \rightarrow \infty} (E'(n)C_1E'(l)(K'\mu)'(E'(l)C_2E'(m)K'^{-1})) \\
 &= \lim_{m, n \rightarrow \infty} E'(n)C_1E'(l)E(m)\mu^C(C_2) \\
 &= \mu^C(C_1C_2)
 \end{aligned}$$

for each $C_1, C_2 \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$, which implies since $\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$ that $(K'\mu)'((\mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2)$ is total in $\mathcal{H}$. We show $J_\mu K \cdot$

$K'^{-1} = S_{K'\mu} = S_{(K'\mu)''}$. Since

(3.3)

$K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is densely contained

in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$,

$$J_\mu K \cdot K'^{-1} K'\mu(Y) = J_\mu K\mu(Y) = K'\mu(Y^\dagger) = S_{K'\mu}(K'\mu)(Y)$$

for each $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$, we have $J_\mu K \cdot K'^{-1} \subset S_{K'\mu}$. We generally have $S_{K'\mu} \subset S_{(K'\mu)'}$, and hence $J_\mu K \cdot K'^{-1} \subset S_{K'\mu} \subset S_{(K'\mu)'}$. Conversely we show $S_{(K'\mu)''} \subset J_\mu K \cdot K'^{-1}$. It is shown similarly to (3.1) that

$$\begin{aligned} E'(n)CE'(m) &\in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\ (K'\mu)'(E'(n)CE'(m)) &= E'(n)KE'(m)\mu^{CCC}(C), \\ &\quad \forall C \in \mathcal{D}(\mu^{CCC})^* \cap \mathcal{D}(\mu^{CCC}), \quad \forall m, n \in \mathbb{N}, \\ J_\mu A^* J_\mu &\in \mathcal{D}(\mu^{CCC}) \text{ and } \mu^{CCC}(J_\mu A^* J_\mu) = \mu^{CC}(A), \\ &\quad \forall A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}), \end{aligned}$$

so that by (3.1)

$$\begin{aligned} E'(n)J_\mu A^* J_\mu E'(m) &\in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\ (K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m)) &= KE(m)E'(n)\mu^{CC}(A), \\ &\quad \forall A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}). \end{aligned}$$

Hence we have

$$\begin{aligned} (KK'^{-1}K'E'(n)E(m)\mu^{CC}(A)|(K'\mu)''(B)) \\ &= ((K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m))|(K'\mu)''(B)) \\ &= (S_{(K'\mu)''}(K'\mu)''(B)|S_{(K'\mu)''}^*(K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m))) \\ &= (S_{(K'\mu)''}(K'\mu)''(B)|(K'\mu)'(E'(m)J_\mu A J_\mu E'(n))) \\ &= (S_{(K'\mu)''}(K'\mu)''(B)|KE(n)E'(m)\mu^{CC}(A^*)) \\ &= (S_{(K'\mu)''}(K'\mu)''(B)|J_\mu K'E'(n)E(m)\mu^{CC}(A)) \\ &= (K'E'(n)E(m)\mu^{CC}(A)|J_\mu S_{(K'\mu)''}(K'\mu)''(B)) \end{aligned}$$

for each $A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC})$ and $B \in \mathcal{D}((K'\mu)'')^* \cap \mathcal{D}((K'\mu)'')$, and further it follows from (3.3) that $\{K'E'(n)E(m)\mu^{CC}(A) ; A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}) \text{ and } m, n \in \mathbb{N}\}$ is total in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, which implies $S_{(K'\mu)''} \subset J_\mu K \cdot K'^{-1}$. Thus we have $S_{K'\mu} = S_{(K'\mu)''} = J_\mu K \cdot K'^{-1}$, and hence

$$(3.4) \quad J_{K'\mu} = J_{(K'\mu)''} = J_\mu \text{ and } \Delta_{K'\mu} = \Delta_{(K'\mu)''} = K \cdot K'^{-1}.$$

It follows from (3.4) that $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system if and only if $K^{it}\mathcal{D} \subset \mathcal{D}$ and $K^{it}YK^{-it}\lceil \mathcal{D} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$. This completes the proof. $\square$

We consider when the condition in Proposition 3.1 $K^{it}\mathcal{D} \subset \mathcal{D}$ and $K^{it}YK^{-it}\lceil \mathcal{D} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$ holds.

Corollary 3.2. *Let $(\mathcal{M}, \mu, K')$ be given in Proposition 3.1. Suppose μ is full, that is, $\mathcal{D}(\mu) = \{X \in \mathcal{M}; \exists \xi_X \in \mathcal{D} \text{ s.t. } X\lambda^C(C) = C\xi_X, \forall C \in \mathcal{D}(\lambda^C)^* \cap \mathcal{D}(\lambda^C)\}$, and $K^{it}\lceil \mathcal{D} \in \mathcal{M}$ for all $t \in \mathbb{R}$. Then $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system.*

Proof. Take arbitrary $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$. Then we have

$$\begin{aligned}
 (K^{it}YK^{-it}\mu^C(C)|\xi) &= (J_\mu K'^{-it}\mu^C(C^*)|Y^\dagger K^{-it}\xi) \\
 &= \lim_\alpha (J_\mu C_\alpha K'^{-it}\mu^C(C^*)|Y^\dagger K^{-it}\xi) \\
 &= \lim_\alpha (Y\mu^C(CK'^{-it}C_\alpha^*)|K^{-it}\xi) \\
 &= \lim_\alpha (CK'^{-it}C_\alpha^*\mu(Y)|K^{-it}\xi) \\
 &= (CK^{it}K'^{-it}\mu(Y)|\xi)
 \end{aligned}$$

for all $C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$ and $\xi \in \mathcal{D}$, where $\{C_\alpha\}$ is a net in $\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$ which converges strongly* to I . Hence it follows from the fullness of μ that $K^{it}YK^{-it} \in \mathcal{D}(\mu)$. By Proposition 3.1 $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system. $\square$

We next consider the converse of Proposition 3.1:

When is a standard system $(\mathcal{M}, \lambda, \lambda')$ unitarily equivalent to such a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$ in Proposition 3.1?

Proposition 3.3. *Let $\mathcal{M}$ be a closed semifinite O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$. Suppose λ is a generalized vector for $\mathcal{M}$ such that*

- (i) $\lambda((\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))^2)$ is total in $\mathcal{H}$;
- (ii) $\lambda'((\mathcal{D}(\lambda')^* \cap \mathcal{D}(\lambda'))^2)$ is total in $\mathcal{H}$;
- (iii) $S_\lambda = S_{\lambda''}$;
- (iv) $\overline{Y} \in L^2(\tau'')$ for each $Y \in \mathcal{D}(\lambda)$, where τ'' is a faithful normal semifinite trace on $(\mathcal{M}'_w)'$.

Then there exist a standard tracial generalized vector μ for a closed O^ -algebra $\mathcal{N}$ in $L^2(\tau'')$ and a non-singular positive self-adjoint operator K' in $L^2(\tau'')$ affiliated with $\mathcal{N}'_w$ such that (μ, K') satisfies all of the conditions in Proposition 3.1 and λ is unitarily equivalent to the generalized vector $K'\mu$; that is, there exists a unitary operator U of $L^2(\tau'')$ onto $\mathcal{H}$ such that $U^* \mathcal{M} U = \mathcal{N}$, $U^* \mathcal{D}(\lambda) U = \mathcal{D}(\mu)$ and $\lambda(Y) = U(K'\mu)(U^* Y U)$ for each $Y \in \mathcal{D}(\lambda)$.*

Proof. By the assumption for λ , $\lambda''(\mathcal{D}(\lambda'')^* \cap \mathcal{D}(\lambda''))$ is an achieved left Hilbert algebra in $\mathcal{H}$ whose left von Neumann algebra equals the semifinite von Neumann algebra $(\mathcal{M}'_w)'$, so that the following results have been shown by Takesaki [15] :

(3.5) We put $\Pi_0 \lambda''(B) = B$, $B \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$. Then Π_0 is a closable operator of the dense subspace $\lambda''(\mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''})$ onto the dense subspace $\mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$ in $L^2(\tau'')$ whose closure Π is non-singular.

(3.6) Let $\Pi = VT'$ be the polar decomposition of Π . Then V is a unitary operator of $\mathcal{H}$ onto $L^2(\tau'')$ and T' is a non-singular positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}'_w$ such that $\Delta_{\lambda''}^{\frac{1}{2}} = T'^{-1} \cdot T'$, where $T = J_{\lambda''} T' J_{\lambda''}$.

(3.7) Let ρ_0 be the left regular representation of $(\mathcal{M}'_w)'$ on $L^2(\tau'')$ defined by $\rho_0(A)B = AB$, $A \in (\mathcal{M}'_w)'$, $B \in \mathfrak{N}_{\tau''}$. Then the unitary operator V implements a spatial isomorphism between $(\mathcal{M}'_w)'$ and $\rho_0((\mathcal{M}'_w)')$ such that $VBV^* = \rho_0(B)$ for each $B \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$.

(3.8) $\lambda''(\mathcal{D}(\lambda'')^* \cap \mathcal{D}(\lambda'')) \cap \mathfrak{N}_{\tau''}$ is dense in the Hilbert space $\mathcal{D}(T')$.

Let Λ be the inverse of Π and $\Lambda = UK'$ be the polar decomposition of Λ . Then we have $U = V^*$ and $K' = U^* T'^{-1} U$. It follows from (3.6) that U is a unitary operator of $L^2(\tau'')$ onto $\mathcal{H}$ and K' is a non-singular positive self-adjoint operator

in $L^2(\tau'')$ affiliated with the von Neumann algebra $\rho_0((\mathcal{M}'_w)')$. We put

$$\begin{aligned}\mathcal{N} &= U^* \mathcal{M} U, \\ \mathcal{D}(\mu) &= U^* \mathcal{D}(\lambda) U \text{ and } \mu(U^* Y U) = \overline{Y} \in L^2(\tau''), \quad Y \in \mathcal{D}(\lambda).\end{aligned}$$

Then $\mathcal{N}$ is a closed O^* -algebra on $U^* \mathcal{D}$ in $L^2(\tau'')$ such that $\mathcal{N}'_w = U^* \mathcal{M}'_w U$ and $(\mathcal{N}'_w)' = U^* (\mathcal{M}'_w)' U = \rho_0((\mathcal{M}'_w)'),$ and by (3.7) μ is a tracial generalized vector for $\mathcal{N}$. We show

$$\lambda(Y) \in \mathcal{D}(\Pi) \text{ and } \Pi \lambda(Y) = \mu(U^* Y U), \quad Y \in \mathcal{D}(\lambda).$$

In fact, let $X \in \mathcal{M}$ and let $\overline{X} = U_X |\overline{X}|$ be the polar decomposition of $\overline{X}$ and $|\overline{X}| = \int_0^\infty t dE_X(t)$ the spectral resolution of $|\overline{X}|$. Take an arbitrary $Y \in \mathcal{D}(\lambda)$. Then it is shown that $\overline{Y} E_Y(n) \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$ and $\lambda''(\overline{Y} E_Y(n)) = E_{Y^\dagger}(n) \lambda(Y)$, $n \in \mathbb{N}$. Hence we have

$$\begin{aligned}\lim_{n \rightarrow \infty} \|E_{Y^\dagger}(n) \lambda(Y) - \lambda(Y)\| &= 0, \\ \lim_{n \rightarrow \infty} \|\Pi E_{Y^\dagger}(n) \lambda(Y) - \mu(U^* Y U)\| &= \lim_{n \rightarrow \infty} \tau''((I - E_Y(n)) |\overline{Y}|^2) = 0,\end{aligned}$$

which implies $\lambda(Y) \in \mathcal{D}(\Pi)$ and $\Pi \lambda(Y) = \mu(U^* Y U)$. Hence we have $\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$. Since $K' \mu(U^* Y U) = U^* \lambda(Y)$ for each $Y \in \mathcal{D}(\lambda)$ and $\lambda((\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))^2)$ is total in $\mathcal{H}$, it follows that $(K' \mu)((\mathcal{D}(K' \mu)^\dagger \cap \mathcal{D}(K' \mu))^2)$ is total in $L^2(\tau'')$. Further, since we have

$$\begin{aligned}(K' \mu)(U^* Y U) &= U^* \lambda(Y), \\ (K \cdot K'^{-1})(K' \mu)(U^* Y U) &= U^* T^{-1} \cdot T' \lambda(Y) = U^* J_\lambda \lambda(Y^\dagger)\end{aligned}$$

for each $Y \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$, and further $S_\lambda = S_{\lambda''} = J_\lambda T^{-1} \cdot T'$ by (3.6) and $\lambda(\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))$ is dense in the Hilbert space $\mathcal{D}(S_{\lambda''}) = \mathcal{D}(T^{-1} \cdot T')$, it follows that $K' \mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$. Thus the pair (μ, K') satisfies all of the conditions in Proposition 3.1. It is clear that $\mathcal{D}(\mu) = U^* \mathcal{D}(\lambda) U$ and $\lambda(Y) = U(K' \mu)(U Y U^*)$ for each $Y \in \mathcal{D}(\lambda)$. This completes the proof. $\square$

By Propositions 3.1, 3.3 we have the following

Theorem 3.4. *Let $\mathcal{M}$ be a closed semifinite O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_w \mathcal{D} \subset \mathcal{D}$, and let λ be a generalized vector for $\mathcal{M}$. The following statements are equivalent:*

- (i) $(\mathcal{M}, \lambda, \lambda')$ is a standard system such that $\overline{Y} \in L^2(\tau'')$ for each $Y \in \mathcal{D}(\lambda)$, where τ'' is a faithful normal semifinite trace on $(\mathcal{M}'_w)'_+$.
- (ii) There exist a closed O^* -algebra $\mathcal{N}$ on $\mathcal{E}$ in $\mathcal{K}$, a standard tracial generalized vector μ for $\mathcal{N}$ and a non-singular positive self-adjoint operator K' in $\mathcal{K}$ affiliated with $\mathcal{N}'_w$ such that
 - (ii)₁ $\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$ and $K' \mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{K}$;
 - (ii)₂ $K' \mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, where $K \equiv J_\mu K' J_\mu$;
 - (ii)₃ $K^{it} \mathcal{E} \subset \mathcal{E}$ and $K^{it} Y K^{-it} [\mathcal{E} \in \mathcal{D}(\mu)]$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$;
 - (ii)₄ λ is unitarily equivalent to the generalized vector $K' \mu$; that is, there exists a unitary operator U of $\mathcal{K}$ onto $\mathcal{H}$ such that $U^* \mathcal{M} U = \mathcal{N}$, $U^* \mathcal{D}(\lambda) U = \mathcal{D}(\mu)$ and $\lambda(Y) = U(K' \mu)(U^* Y U)$ for each $Y \in \mathcal{D}(\lambda)$.

Corollary 3.5. *Let $\mathcal{M}$ be an integrable O^* -algebra on $\mathcal{D}$ in $\mathcal{H}$ with a standard tracial vector ξ_0 , and let $\xi \in \mathcal{D}$. Then ξ is a standard vector for $\mathcal{M}$ if and only if there exists a non-singular positive self-adjoint operator K' in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\text{w}}$ such that (a) $\xi_0 \in \mathcal{D}(K')$ and $K'\xi_0 \in \mathcal{D}$; (b) $\mathcal{M}\xi_0$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, where $K \equiv J_{\xi_0}K'J_{\xi_0}$; (c) $K^{it}[\mathcal{D} \in \mathcal{M}$ for each $t \in \mathbb{R}$; (d) ξ is unitarily equivalent to $K'\xi_0$.*

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