A CHARACTERIZATION OF THE LEINERT PROPERTY

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ABSTRACT. Let G be a discrete group and denote by λ_G its left regular representation on $\ell_2(G)$. Denote further by $\mathbf{F}_n$ the free group on n generators $\{g_1, g_2, \dots, g_n\}$ and λ its left regular representation. In this paper we show that a subset $S = \{t_1, t_2, \dots, t_n\}$ of G has the Leinert property if and only if for some real positive coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ the identity

$$\left\| \sum_{i=1}^n \alpha_i \lambda_G(t_i) \right\|_{C_\lambda^*(G)} = \left\| \sum_{i=1}^n \alpha_i \lambda(g_i) \right\|_{C_\lambda^*(\mathbf{F}_n)}$$

holds. Using the same method we obtain some metric estimates about abstract unitaries $U_1, U_2, \dots, U_n$ satisfying the similar identity $\left\| \sum_{i=1}^n U_i \otimes \overline{U_i} \right\|_{\min} = 2\sqrt{n-1}$.

Notation. Throughout in this paper G will denote a discrete group with unit element e and $C_\lambda^*(G)$ the sub- C^* -algebra of $B(\ell_2(G))$ generated by its left regular representation λ_G . This algebra is equipped with the trace state $\tau_G(X) = \langle X\delta_e, \delta_e \rangle$. A subset $\{t_1, t_2, \dots, t_n\}$ of G is called *free* if it generates a copy of $\mathbf{F}_n$, the free group on n generators. We shall denote the canonical generators of the free group by $\{g_1, g_2, \dots, g_n\}$. When considering the free group we shall omit the subscript in $\lambda_{\mathbf{F}_n}$. We shall almost exclusively deal with finitely generated groups and repeatedly use the fact that given a generating set $\{h_1, h_2, \dots, h_n\}$ for the group G there is a (unique) quotient mapping $q: \mathbf{F}_n \rightarrow G$ with $q(g_i) = h_i \quad \forall i \in \{1, 2, \dots, n\}$.

The following proposition collects some results from [K, Lemma 3.1 and Theorem 3]:

Proposition 1. *Let $\{h_1, h_2, \dots, h_n\}$ be a generating set of the group G and denote by $\alpha_1, \alpha_2, \dots, \alpha_n$ positive real numbers. Then*

$$\left\| \sum_{i=1}^n \alpha_i (\lambda_G(h_i) + \lambda_G(h_i)^*) \right\|_{C_\lambda^*(G)} \geq \left\| \sum_{i=1}^n \alpha_i (\lambda(g_i) + \lambda(g_i)^*) \right\|_{C_\lambda^*(\mathbf{F}_n)}.$$

Moreover in the case of equal coefficients,

$$\left\| \sum_{i=1}^n (\lambda_G(h_i) + \lambda_G(h_i)^*) \right\|_{C_\lambda^*(G)} = \left\| \sum_{i=1}^n (\lambda(g_i) + \lambda(g_i)^*) \right\|_{C_\lambda^*(\mathbf{F}_n)} = 2\sqrt{2n-1}$$

if and only if $\{h_1, h_2, \dots, h_n\}$ is a free set.

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This result has a graph-theoretical interpretation, where the operator in consideration corresponds to the combinatorial Laplacian on the Cayley graph of the group G ; see [Pa] in particular for a generalization of the second statement. However it remained unclear whether the free group can be characterized by the norms of non-selfadjoint operators, and this question will be the subject of this paper. The following proposition is an extension of [K, Lemma 3.1].

Proposition 2. *Let G and H be discrete groups and $q : G \rightarrow H$ a group homomorphism. Then for every finite subset $\{t_1, t_2, \dots, t_m\}$ of G and corresponding nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ the inequality*

$$(1) \quad \left\| \sum_{i=1}^m \alpha_i \lambda_H(q(t_i)) \right\|_{C_\lambda^*(H)} \geq \left\| \sum_{i=1}^m \alpha_i \lambda_G(t_i) \right\|_{C_\lambda^*(G)}$$

holds.

Proof. We will use the following well known fact about the noncommutative L^p -norms associated to τ_G :

$$\|X\|_{C_\lambda^*(G)} = \sup_{1 \leq p < \infty} (\tau_G((X^*X)^p))^{1/2p}.$$

It suffices to consider the integer values of p . Denote by $W_p^{alt}(A)$ the set of all alternating words of length $2p$ in the letters $A = \{t_1, t_2, \dots, t_m\}$:

$$W_p^{alt}(A) = \left\{ t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} : i_1, j_1, i_2, j_2, \dots, i_p, j_p \in \{1, 2, \dots, m\} \right\}.$$

For $v = t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} \in W_p^{alt}(A)$ we use the following abbreviation:

$$(2) \quad \alpha_v = \alpha_{i_1} \alpha_{j_1} \alpha_{i_2} \alpha_{j_2} \cdots \alpha_{i_p} \alpha_{j_p}.$$

Then we can write (note that we identify words in $W_p^{alt}(A)$ with the corresponding elements in the group G):

$$\begin{aligned} \left\| \sum_{i=1}^m \alpha_i \lambda_H(q(t_i)) \right\|_{C_\lambda^*(H)} &= \sup_p \left(\tau_H \left(\sum_{i,j=1}^m \alpha_i \alpha_j \lambda_H(q(t_i^{-1} t_j)) \right)^p \right)^{1/2p} \\ &= \sup_p \left(\sum_{v \in W_p^{alt}(A)} \alpha_v \tau_H(\lambda_H(q(v))) \right)^{1/2p} = \sup_p \left(\sum_{\substack{v \in W_p^{alt}(A) \\ q(v)=e}} \alpha_v \right)^{1/2p} \\ &\geq \sup_p \left(\sum_{\substack{v \in W_p^{alt}(A) \\ v \stackrel{G}{=} e}} \alpha_v \right)^{1/2p} = \left\| \sum_{i=1}^m \alpha_i \lambda_G(t_i) \right\|_{C_\lambda^*(G)}. \end{aligned}$$

□

In [P2] the following more general statement is proved:

Proposition 3. *Let H be a Hilbert space and $U_1, U_2, \dots, U_n$ a finite sequence of unitary operators acting on H . Consider the representation π of $\mathbf{F}_n$ which is determined by the condition $\pi(g_i) = U_i$ for all $i \in \{1, 2, \dots, n\}$. Then for every set of words $\{w_1, w_2, \dots, w_m\} \in \mathbf{F}_n$ and every sequence $\alpha_1, \alpha_2, \dots, \alpha_m$ of positive real numbers we have*

$$\left\| \sum_{i=1}^m \alpha_i \pi(w_i) \otimes \overline{\pi(w_i)} \right\|_{\min} \geq \left\| \sum_{i=1}^m \alpha_i \lambda(w_i) \right\|_{C_\lambda^*(\mathbf{F}_n)}.$$

The set $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}\}$ is only a special case of a *Leinert set*, a concept which appeared first in [Le].

Definition 4 ([A-O, Definition III B and Theorem III D]).

A subset $A = \{t_1, t_2, \dots, t_n\}$ of a discrete group G with unit element e is called a *Leinert set* if it satisfies one of the following equivalent conditions:

- Every sequence $t_{i_1}, t_{j_1}, t_{i_2}, t_{j_2}, \dots, t_{i_m}, t_{j_m}$ with $i_k, j_k \in \{1, 2, \dots, n\}$ such that $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ satisfies

$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} \neq e.$$

- The set A can be written as $y(B \cup \{e\})$, where B is a free subset of G and $y \in G$.

The next proposition extends Kesten's formula for the norm (see also [B] for a weaker version). Simplified proofs can be found in [P-P] and [W]. For recent examples of Leinert sets in one-relator groups see [C-V].

Proposition 5 ([A-O, Theorem IV J]). *Let $n \geq 2$ and $\{t_1, \dots, t_n\}$ be a Leinert set in a discrete group G . Then*

$$\left\| \sum_{i=1}^n \lambda_G(t_i) \right\|_{C_\lambda^*(G)} = 2\sqrt{n-1}.$$

The next proposition is standard.

Proposition 6. *If $u_1, u_2, \dots, u_n$ are unitaries in a C^* -algebra A for which there exist positive real numbers $\beta_1, \beta_2, \dots, \beta_n$ satisfying*

$$\left\| \sum_{i=1}^n \beta_i u_i \right\| = \sum_{i=1}^n \beta_i$$

then there exists a state φ of A such that for every $m \in \mathbb{N}$

$$\varphi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1 \quad \forall i_k, j_k = 1, 2, \dots, n.$$

In particular, if $t_1, t_2, \dots, t_n$ are some elements of the group G then the subgroup of G generated by $\{t_i^{-1} t_j : i, j = 1, 2, \dots, n\}$ is amenable if and only if $\|\sum \beta_i \lambda_G(t_i)\| = \sum \beta_i$ for some positive coefficients $\beta_1, \beta_2, \dots, \beta_n$.

Proof. By a compactness argument, there is a state on A such that $\varphi(\sum \beta_i \beta_j u_i^* u_j) = \|\sum \beta_i \beta_j u_i^* u_j\| = \sum \beta_i \beta_j$. It follows that $\varphi(u_i^* u_j) = 1$ for all $i, j = 1, 2, \dots, n$ and by induction on m we have $\phi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1$ for any choice of indices i_k, j_k . Indeed,

$$\begin{aligned} & \left| \phi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m} - u_{i_m}^* u_{j_m} u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) \right| \\ & \leq \phi((I - u_{i_m}^* u_{j_m})^* (I - u_{i_m}^* u_{j_m}))^{1/2} = 0. \end{aligned}$$

Now in the case where $u_i = \lambda_G(t_i)$ this implies that the trivial representation of the group generated by $t_i^{-1}t_j$ extends to a representation of the reduced group C^* -algebra, which implies that it is amenable. $\square$

The following lemma and its corollary will play a key rôle in the generalization of Kesten's result.

Lemma 7 ([H-R-V1, Lemma 8]). *Let μ be a probability measure on $\mathbf{R}$ with compact support and suppose*

$$\begin{aligned} -m &= \min \operatorname{supp} \mu < \max \operatorname{supp} \mu = M, \\ \mu_n &= \int_{\mathbf{R}} t^n d\mu(t) \geq 0 \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Then

$$\limsup_{n \rightarrow \infty} (\mu_n)^{1/n} = M = \max(m, M).$$

Corollary 8 ([K, Lemma 3.2]). *For every subset $\{t_1, t_2, \dots, t_n\}$ of G and every sequence of positive real numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ we have*

$$\left\| \alpha_0 I + \sum_{i=1}^n \alpha_i (\lambda_G(t_i) + \lambda_G(t_i)^*) \right\| = \alpha_0 + \left\| \sum_{i=1}^n \alpha_i (\lambda_G(t_i) + \lambda_G(t_i)^*) \right\|$$

and a similar result holds for general unitaries:

$$\left\| \alpha_0 I + \sum_{i=1}^n \alpha_i (U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}) \right\| = \alpha_0 + \left\| \sum_{i=1}^n \alpha_i (U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}) \right\|.$$

Proof. This follows from Lemma 7 applied to the probability measure μ on the spectrum of the self-adjoint operator

$$T = \sum_{i=1}^n \alpha_i (\lambda(t_i) + \lambda(t_i)^*)$$

which is determined by the moments

$$\mu_n = \tau_G(T^n).$$

For the case of general unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$ we refer to [P1, Example 5.6] where the following formula is proved: For any finite sequence of operators $x_1, x_2, \dots, x_n$ on some Hilbert space H we have

$$(3) \quad \left\| \sum x_i \otimes \overline{x_i} \right\|_{\min} = \sup_{y, z \in (S_2^+)_1} \sum \operatorname{tr}(x_i y x_i^* z)$$

where $(S_2^+)_1$ is the intersection of the unit ball of the Hilbert-Schmidt class operators on H with the cone of positive operators. In our case the operator is self-adjoint and we can restrict (3) to the symmetric part:

$$\begin{aligned} \left\| \alpha_0 I + \sum_{i=1}^n \alpha_i (\hat{U}_i + \hat{U}_i^*) \right\|_{\min} &= \sup_{y \in (S_2^+)_1} \operatorname{tr} \left(\alpha_0 y^2 + 2 \sum_{i=1}^n \hat{U}_i y \hat{U}_i^* y \right) \\ &= \alpha_0 + \left\| \sum_{i=1}^n \alpha_i (\hat{U}_i + \hat{U}_i^*) \right\|_{\min}. \end{aligned}$$

$\square$

Now we are ready to prove the main result.

Theorem 9. *Let G be a discrete group, $n \geq 3$ an integer and let $t_1, t_2, \dots, t_n$ be some elements in G . Then for any sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ of strictly positive numbers we have*

$$\left\| \sum_{i=1}^n \alpha_i \lambda_G(t_i) \right\|_{C_\lambda^*(G)} = \left\| \sum_{i=1}^n \alpha_i \lambda(g_i) \right\|_{C_\lambda^*(\mathbf{F}_n)}$$

if and only if $\{t_1, t_2, \dots, t_n\}$ is a Leinert set. In particular, this is equivalent to the identity

$$\left\| \sum_{i=1}^n \lambda_G(t_i) \right\|_{C_\lambda^*(G)} = 2\sqrt{n-1}.$$

Proof. We only have to show the “only if” part and we shall use the method introduced in [K, Theorem 3] (see also [B], where similar methods are used. We are grateful to A. Valette for bringing this article to our attention). Let us compare the norms of the operators

$$\tilde{T} = \sum_{i=1}^n \alpha_i \lambda_G(t_i) \quad \text{and} \quad T = \sum_{i=1}^n \alpha_i \lambda(g_i).$$

We can assume that G is generated by the subset $\{t_1, t_2, \dots, t_n\}$. Suppose that the set $\{t_1, t_2, \dots, t_n\}$ does not have the Leinert property. This means that there are an integer m and a sequence of indices $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$, with $i_k, j_k \in \{1, 2, \dots, n\}$ and such that

$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} = e.$$

Let $g_1, g_2, \dots, g_n$ be the generators of the free group $\mathbf{F}_n$ and let $q : \mathbf{F}_n \rightarrow G$ be the quotient mapping associated to the generating set $\{t_1, t_2, \dots, t_n\}$. Taking the same indices as above, set

$$w := g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}.$$

Since $\{g_1, g_2, \dots, g_n\}$ is a Leinert set, the word w does not reduce to the identity in $\mathbf{F}_n$ but it is in the kernel of q . We can assume without loss of generality that w is cyclically reduced, i.e. $j_m \neq i_1$. For if this is not the case, we can consider the word

$$g_{i_m}^{-1} g_{i_1} w g_{i_1}^{-1} g_{i_m} = g_{i_m}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_{m-1}}^{-1} g_{j_{m-1}}$$

which is cyclically reduced and still in the kernel of q . Note that it cannot be reduced to the empty word in this way, since the conjugacy class of the latter in $\mathbf{F}_n$ is trivial.

So we can assume for the sake of clarity that $i_1 = 1$ and $j_m = 2$. Now consider the elements

$$\begin{aligned} w' &= g_1^{-1} g_3 w g_3^{-1} g_1, \\ w'' &= g_2^{-1} g_3 w g_3^{-1} g_2 \end{aligned}$$

which are both in the kernel of q and have infinite order, because w has, being cyclically reduced. Moreover, they form a free set, since there are no possible

cancellations in non-trivial reduced words built out of them. Thus we can apply Proposition 6 to the operator

$$T_1 = \alpha_{w'} (\lambda(w') + \lambda(w')^*) + \alpha_{w''} (\lambda(w'') + \lambda(w'')^*)$$

and we get

$$(4) \quad \|T_1\| < 2(\alpha_{w'} + \alpha_{w''})$$

because the group generated by $\{w'^{-1}w'', w'^2, w''^2\}$ is not amenable.

To compare the norms of T and $\tilde{T}$ we use the identity

$$\|X\|^2 = \|X^*X\| = \|(X^*X)^p\|^{1/p},$$

which is a consequence of Gel'fand's theorem. Thus it suffices to show that for some p

$$\|(T^*T)^p\| < \|(\tilde{T}^*\tilde{T})^p\|.$$

We consider $p = m + 2$ so that $2p = |w'| = |w''|$. Denote by $W_p^{alt}(n)$ the set of all alternating words of length $2p$ in n generators:

$$W_p^{alt}(n) = \{g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_p}^{-1}g_{j_p} : i_1, j_1, i_2, j_2, \dots, i_p, j_p \in \{1, 2, \dots, n\}\}.$$

Using the notation of (2) we can write

$$\begin{aligned} (T^*T)^{m+2} &= \sum_{v \in W_{m+2}^{alt}(n)} \alpha_v \lambda(v) \\ &= T_0 + T_1 \end{aligned}$$

where

$$T_0 = \sum_{v \in W_{m+2}^{alt}(n) \setminus \{w', w'^{-1}, w'', w''^{-1}\}} \alpha_v \lambda(v)$$

and T_1 is the same as above. Obviously, setting

$$\tilde{T}_0 = \sum_{v \in W_{m+2}^{alt}(n) \setminus \{w', w'^{-1}, w'', w''^{-1}\}} \alpha_v \lambda_G(q(v))$$

we have

$$(\tilde{T}^*\tilde{T})^{m+2} = \tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I.$$

Observe that T_0 (and hence $\tilde{T}_0$) is selfadjoint with positive coefficients so that we can apply (1) and Corollary 8 and we get

$$\begin{aligned} \|(T^*T)^{m+2}\| &\leq \|T_0\| + \|T_1\| < \|\tilde{T}_0\| + 2(\alpha_{w'} + \alpha_{w''}) \\ &= \|\tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I\| = \|(\tilde{T}^*\tilde{T})^{m+2}\|. \end{aligned}$$

□

Remark. Actually in the case of equal coefficients the above proof gives the following quantitative estimate (cf. [K, (4.15)] and [Pa, Theorem 3.2]): With the notation as in the proof, let m denote the minimal length of a nontrivial word in the kernel of the quotient mapping q . Then

$$\|\tilde{T}\| \geq \|T\| + \frac{2(n-1) - 2\sqrt{2n-3}}{(2m+4)n^{2m+3}}.$$

Indeed, when there are n free generators, we can build a free set $\{w_1, w_2, \dots, w_{n-1}\}$ in $\mathbf{F}_n$ by setting $w_i = g_i^{-1} g_n w g_n^{-1} g_i$ for $i \in \{1, 2, \dots, n-1\}$. Then

$$\left\| \sum_{i=1}^{n-1} (\lambda(w_i) + \lambda(w_i)^*) \right\| = 2\sqrt{2n-3}$$

and by modifying T_0 and T_1 (resp. $\tilde{T}_0$ and $\tilde{T}_1$ appropriately), we have the inequality

$$\|\tilde{T}\|^{2(m+2)} \geq \|T\|^{2(m+2)} + 2(n-1) - 2\sqrt{2n-3}.$$

Now a simple convexity argument and the fact that $\|\tilde{T}\| \leq n$ yield the claim.

The following corollary gives a characterization of the free group in terms of the norms operators in the reduced C^* -algebra. Note that the free group cannot be characterized by the spectrum of the sum of the generators, as shown in [H-R-V, sect.4].

Corollary 10. *Let G be a discrete group and $S = \{t_0 = e, t_1, \dots, t_n\}$ be a generating subset with $n \geq 2$. Then*

$$2\sqrt{n} \leq \left\| \sum_{i=0}^n \lambda_G(t_i) \right\|$$

with equality if and only if G is the free group on n generators.

A similar proof yields the following result about unitaries which do not necessarily arise from a regular representation of a discrete group, as for example in [L-P-S].

Theorem 11. *Let $U_1, \dots, U_n$ be unitary operators acting on some Hilbert space H and suppose*

$$\left\| \sum_{i=1}^n U_i \otimes \overline{U_i} \right\|_{\min} = 2\sqrt{n-1}.$$

Then for all products of the form $V = U_{i_1}^{-1} U_{j_1} U_{i_2}^{-1} U_{j_2} \cdots U_{i_m}^{-1} U_{j_m}$ with $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ we have

$$\|V \otimes \overline{V} - I\| \geq 1.$$

Proof. To simplify notation we use the unitary representation

$$\hat{\pi} = \pi \otimes \overline{\pi} : \mathbf{F}_n \rightarrow B(H \otimes_2 \overline{H})$$

which we already introduced in Proposition 3 and which maps the generators g_i to the corresponding unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$. Now let

$$v_0 = g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}$$

be a word as in the claim and set $\hat{V} = \hat{\pi}(v_0) = V \otimes \overline{V}$. We can assume v_0 to be cyclically reduced, because $\|\hat{V} - I\| = \|W\hat{V}W^{-1} - I\|$ for any unitary W and in the case v_0 is not cyclically reduced, we can take a cyclically reduced one in its conjugacy class. We can of course assume that $i_1 = 1$ and $j_m = 2$. Now for the cyclically reduced word v_0 the set

$$F_k = \{(g_1^{-1} g_3)^r v_0 (g_1^{-1} g_3)^{-r} : r = 1, 2, \dots, k\}$$

is a free set and we can do the same estimate as in the above proof. Setting $T = \sum_{i=1}^n \lambda(g_i)$ as above and $\hat{T} = \sum_{i=1}^n \hat{U}_i$ we can identify F_k with some subset of the set

of unreduced words $W_{m+2k}^{alt}(n)$, e.g. the set $\{(g_1^{-1}g_3)^r v_0 (g_1^{-1}g_3)^{-r} (g_1^{-1}g_1)^{k-r} : r = 1, 2, \dots, k\}$, and get

$$\begin{aligned} \|(T^*T)^{m+2k}\| &= \left\| \sum_{v \in W_{m+2k}^{alt}(n)} \lambda(v) \right\| \\ &\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \lambda(v) \right\| + \left\| \sum_{v \in F_k} \lambda(v) + \lambda(v)^* \right\| \\ &\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \hat{\pi}(v) \right\| + 2\sqrt{2k-1}. \end{aligned}$$

The last inequality follows from (1) and Proposition 5. We can now apply Corollary 8 to the first term on the right-hand side and we obtain

$$\begin{aligned} \|(T^*T)^{m+2k}\| &\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n) \setminus (F_k \cup F_k^{-1})} \hat{\pi}(v) + \sum_{v \in F_k} 2I \right\| + 2\sqrt{2k-1} - 2k \\ &\leq \left\| \sum_{v \in W_{m+2k}^{alt}(n)} \hat{\pi}(v) \right\| + \left\| \sum_{v \in F_k} 2I - \hat{\pi}(v) - \hat{\pi}(v)^* \right\| + 2\sqrt{2k-1} - 2k \\ &\leq \|(\hat{T}^*\hat{T})^{m+2k}\| + 2k\|\hat{\pi}(v_0) - I\| + 2\sqrt{2k-1} - 2k. \end{aligned}$$

This together with the assumption $\|T\| = \|\hat{T}\|$ yields

$$\|\hat{V} - I\| \geq 1 - \frac{\sqrt{2k-1}}{k}$$

and since this holds for all k , the proof is complete. $\square$

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NOTE ADDED IN PROOF

We noticed a connection to the paper of V. Flory, “Estimating norms in C^* -algebras of discrete groups”, Math. Ann. **224** (1976) 41–52. Using our results Theorem 8 in the latter paper can be sharpened for finite sets as follows.

Let G be a discrete group and $E \subset G$ a finite subset. Theorem 8 can be sharpened for finite subsets as follows. Define for $K \subset G$ the *Leptin constant* $\omega(K) = \inf_{\emptyset \neq U \subset G \text{ finite}} \frac{\#(KU)}{\#(U)}$. Then the following are equivalent.

- (1) E has the Leinert property.
- (2) For every $y \in E$ the set $\{y^{-1}x : x \in E \text{ and } x \neq y\}$ is a free subset of G .
- (3) $\omega(E) = \#(E) - 1$.

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