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A CHARACTERIZATION OF RIEMANNIAN FLOWS

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Abstract. We prove that a flow on a closed manifold is Riemannian if and only if it is locally generated by Killing vector fields for a Riemannian metric.

Consider a flow $\mathcal{F}$, i.e. an oriented one-dimensional foliation on a manifold M. The purpose of this note is to prove the following result.

Theorem. *Let M be a closed manifold. Then the flow $\mathcal{F}$ is Riemannian if and only if the tangent bundle of $\mathcal{F}$ is locally generated by Killing vector fields for a Riemannian metric g on M.*

If the flow $\mathcal{F}$ is locally generated by Killing vector fields for a metric g on M, then the flow is clearly Riemannian, and the metric g bundle-like for $\mathcal{F}$ in the sense of Reinhart [8]. What we wish to show is that on a closed manifold the converse also holds.

It is well-known that a Riemannian flow on a closed manifold M^{n+1} is not necessarily defined by a global Killing vector field for a Riemannian metric on M. According to a result of Molino and Sergiescu [7], a necessary and sufficient condition for this to be the case is that $H^n_B(\mathcal{F}) \neq 0$, where $H^n_B(\mathcal{F})$ is the top-dimensional basic cohomology vector space. Carrière [1] proved a general structure theorem for Riemannian flows, and also provided an example of a Riemannian flow on a closed oriented manifold M^3 with $H^2_B(\mathcal{F}) = 0$.

Proof of the theorem. Let $\mathcal{F}$ be a Riemannian flow on the closed manifold M. The fundamental new fact we are going to use is the result of Domínguez [2], establishing the existence of a bundle-like metric g for which the mean curvature 1-form κ is basic. The mean curvature 1-form is dual to the mean curvature vector field, and as such vanishes on vectors tangent to $\mathcal{F}$. The essential property making κ basic is that the Lie derivative $\theta(V)\kappa = 0$ for vector fields V tangent to $\mathcal{F}$. A fact pointed out in [4, (4.4)] is that under this condition $d\kappa = 0$ (see also the proof in [9, (12.5)]). It follows that locally $\kappa = df$. This local function f is necessarily basic, as follows from

$$Vf = df(V) = \kappa(V) = 0$$

for a vector field V tangent to $\mathcal{F}$. It suffices to verify that for such a local unit vector field V the modified $e^{-f}V$ is a local Killing vector field for g.

The property to verify is that $\theta(e^{-f}V)g = 0$. We evaluate this bilinear form successively on the tangent bundle L of $\mathcal{F}$, its g-orthogonal complement $L^\perp$, and

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on mixed arguments. Obviously $(\theta(e^{-f}V)g)(V,V)=0$. To evaluate $\theta(e^{-f}V)g$ on $L^{\perp}$, it suffices to consider projectable local sections X, Y of $L^{\perp}$, since they locally generate all sections of $L^{\perp}$. Then

$$(\theta(e^{-f}V)g)(X,Y)=e^{-f}Vg(X,Y)-g([e^{-f}V,X],Y)-g(X,[e^{-f}V,Y]).$$

The first term on the RHS vanishes, since $g(X,Y)$ is a basic function for bundle-like g. For projectable X, the bracket $[e^{-f}V,X]$ is a tangential vector field, and thus the second term vanishes. Similarly the third term vanishes for a projectable Y. It remains to establish

$$(\theta(e^{-f}V)g)(V,X)=0$$

for projectable X. Now

$$\begin{aligned}(\theta(e^{-f}V)g)(V,X)&=-g([e^{-f}V,V],X)-g(V,[e^{-f}V,X])\\&=-g(V,e^{-f}[V,X]-Xe^{-f}\cdot V)=-e^{-f}g(V,[V,X])-e^{-f}Xfg(V,V).\end{aligned}$$

For V of unit length the mean curvature vector field τ can be expressed by the formula $\tau=\nabla^M_V V$ ([9, (10.3)]), where ∇^M denotes the Levi-Civita connection of g. Moreover we have $g(V,[V,X])=g(V,\nabla^M_V X)$, since $g(V,\nabla^M_X V)=\frac{1}{2}Xg(V,V)=0$. It follows that

$$\begin{aligned}(\theta(e^{-f}V)g)(V,X)&=e^{-f}\{g(\nabla^M_V V,X)-Xf\}\\&=e^{-f}\{g(\tau,X)-df(X)\}=e^{-f}\{\kappa(X)-df(X)\}=0.\end{aligned}\qquad\square$$

For some of the preceding calculations, see also [5]. The argument given shows further that on a simply connected manifold, or more generally on a manifold with finite fundamental group, a Riemannian flow defined by a nonsingular vector field can be defined by a global Killing vector field for a Riemannian metric. This follows from the injectivity of $H^1_B(\mathcal{F})\to H^1_{DR}(M)$ ([9, (9.9)]), which implies that $[\kappa]=0\in H^1_B(\mathcal{F})$, so that the equation $\kappa=df$ is in this case globally solvable with a basic function f. The proof above shows that after normalization the vector field $e^{-f}V$ is such a global Killing vector field for a metric g on M with κ a basic 1-form. But using [1, p. 49, Prop. 1], this shows that the flow is then also geodesible (for a metric renormalized along the leaves, see also [9, (10.9)]).

We end with the following two remarks. According to Carrière [1], the Riemannian flow property of $\mathcal{F}$ is incompatible with the existence of any strictly negative curvature metric on the closed manifold M. On the other hand, by [3], [6], the existence of a strictly positively curved bundle-like metric for $\mathcal{F}$ implies the vanishing of $H^1_B(\mathcal{F})$, and thus by the argument above also the generation of $\mathcal{F}$ by a global Killing vector field for a Riemannian metric on M.

References

1. Y. Carrière, *Flots Riemanniens, Structure transverse des feuilletages*, Astérisque **116** (1984), 31–52. MR **86m:**58125a
2. D. Domínguez, *A tenseness theorem for Riemannian foliations*, C. R. Acad. Sci. Paris **320** (1995), 1331–1335. MR **96d:**53025
3. J. Hebda, *Curvature and focal points in Riemannian foliations*, Indiana Univ. Math. J. **35** (1986), 321–331. MR **87g:**53051
4. F. Kamber and Ph. Tondeur, *Foliations and metrics*, Progress in Math. **32** (1983), 103–152. MR **85a:**57017
5. H. Kyung Pak, *On one-dimensional metric foliations in Einstein spaces*, Ill. J. Math. **36** (1992), 594–599. MR **94d:**53046

6. M. Min-Oo, E. Ruh and Ph. Tondeur, *Vanishing theorems for the basic cohomology of Riemannian foliations*, J. Reine Angew. Math. **415** (1991), 167–174. MR **92b:**58218
7. P. Molino and V. Sergiescu, *Deux remarques sur les flots riemanniens*, Manuscripta Math. **51** (1985), 145–161. MR **86h:**53035
8. B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69** (1959), 119–132. MR **21:**6004
9. Ph. Tondeur, *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag, New York, 1988. MR **89e:**53052

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