S^2 -BUNDLES OVER ASPHERICAL SURFACES AND 4-DIMENSIONAL GEOMETRIES

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(Communicated by Ronald A. Fintushel)

ABSTRACT. Melvin has shown that closed 4-manifolds that arise as S^2 -bundles over closed, connected aspherical surfaces are classified up to diffeomorphism by the Stiefel-Whitney classes of the associated bundles. We show that each such 4-manifold admits one of the geometries $S^2 \times E^2$ or $S^2 \times \mathbb{H}^2$ [depending on whether $\chi(M) = 0$ or $\chi(M) < 0$]. Conversely a geometric closed, connected 4-manifold M of type $S^2 \times E^2$ or $S^2 \times \mathbb{H}^2$ is the total space of an S^2 -bundle over a closed, connected aspherical surface precisely when its fundamental group $\Pi_1(M)$ is torsion free. Furthermore the total spaces of $\mathbb{RP}^2$ -bundles over closed, connected aspherical surfaces are all geometric. Conversely a geometric closed, connected 4-manifold M' is the total space of an $\mathbb{RP}^2$ -bundle if and only if $\Pi_1(M') \cong \mathbb{Z}/2\mathbb{Z} \times K$ where K is torsion free.

INTRODUCTION

Melvin [Me] has shown that closed 4-manifolds that arise as S^2 -bundles over closed aspherical surfaces are classified up to diffeomorphism by the Stiefel-Whitney classes of the associated bundles. Moreover one may construct ξ from $w(\xi)$. See [Me, Structure Lemma]. In particular there are two orientable 4-manifolds that arise as total spaces of S^2 -bundles over each closed surface and are distinguished by whether or not $w_2(\xi) = 0$. Ue [Ue] has shown that the total spaces M of such orientable bundles over the surfaces of Euler characteristic $\chi = 0$ have the structures of Seifert 4-manifolds and are geometric.

There are just two S^2 -bundles over S^2 , the product bundle and $S^2 \tilde{\times} S^2$; the latter is not geometric. There are four S^2 -bundles over $\mathbb{RP}^2$, each of which admits an $S^2 \times S^2$ geometry and is distinguishable by its Stiefel-Whitney classes [H1]. There are four S^2 -bundles over each closed orientable surface B with Euler characteristic $\chi(B) \leq 0$, six S^2 -bundles over the Klein bottle Kb and eight S^2 -bundles over each closed non-orientable surface B with $\chi(B) \leq 0$, [Me]. The aim of this paper is to prove the following:

Main Theorem. (1) *A closed $S^2 \times E^2$ - or $S^2 \times \mathbb{H}^2$ -manifold M is the total space of an S^2 -bundle over some aspherical surface precisely when its fundamental group $\Pi_1(M)$ is torsion free. Conversely all total spaces of such bundles are geometric.*

Received by the editors May 10, 1996.

1991 *Mathematics Subject Classification.* Primary 57N50; Secondary 57N13, 55R25.

Key words and phrases. Aspherical surface, S^2 -bundle, 4-dimensional geometry, Stiefel-Whitney class.

(2) A closed $S^2 \times E^2$ - or $S^2 \times \mathbb{H}^2$ -manifold M' is the total space of an $\mathbb{RP}^2$ -bundle over some aspherical surface precisely when its fundamental group splits as $\Pi_1(M') = \mathbb{Z}/2\mathbb{Z} \times K$, where K is torsion free. Conversely all total spaces of such bundles are geometric.

GENERALITIES

Bundles ξ will be denoted by $p: M \rightarrow B$ and will have fibre the 2-sphere S^2 throughout unless otherwise stated. Let E^2 denote the 2-dimensional Euclidean geometry with underlying space $\mathbb{R}^2$. Similarly let $\mathbb{H}^2$ denote the geometry of the hyperbolic plane and $\mathbb{X}^2$ denote either of the above two geometries with underlying space X^2 , in arguments that encompass both. Henceforth a manifold referred to as being closed is a compact, connected manifold without boundary.

From the long exact homotopy sequence for a fibration we have that $\Pi_1(M) \cong \Pi_1(B)$; in particular $H_1(M) \cong H_1(B)$ with any coefficients. Also, as p is a fibration, the Euler characteristic $\chi(M) = \chi(S^2)\chi(B) = 2\chi(B)$, so by Poincaré duality with coefficients in $\mathbb{Z}/2\mathbb{Z}$ we obtain $\beta_2(M) = 2$ and therefore $H_2(M; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In fact $H_2(M; \mathbb{Z}/2\mathbb{Z})$ is generated by the images of a cross-section $\sigma_*(B)$ and the inclusion of the fibre $i_*(S^2)$; as the fibre has zero self intersection and the intersection number of a cross-section and a fibre is 1, they are linearly independent. We shall explicitly construct such cross-sections where needed.

Let ξ be an S^2 -bundle. The homomorphism $p^*: H^*(B; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^*(M; \mathbb{Z}/2\mathbb{Z})$, induced by p , is injective and the Whitney sum formula gives:

- (a) $w_1(M) = p^*(w_1(B) + w_1(\xi))$, and
- (b) $w_2(M) = p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi))$.

See Melvin [Me], Lemma 1.

The Wu class $v_2(M)$ is characterised by:

$$v_2(M) \smile x = x \smile x, \quad \forall x \in H^2(M; \mathbb{Z}/2\mathbb{Z}).$$

Let $\mathcal{D}: H^2(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H_2(M; \mathbb{Z}/2\mathbb{Z})$ be the Poincaré duality isomorphism. Then we may dualize the above equation in terms of intersections of surfaces by:

$$\mathcal{D}(x) \cdot \mathcal{D}(v_2(M)) \equiv \mathcal{D}(x) \cdot \mathcal{D}(x) \pmod{2}, \quad \forall \mathcal{D}(x) \in H_2(M),$$

so that $\mathcal{D}(v_2(M))$ is the characteristic element for the intersection form. Together with the Wu relation $v_2(M) = w_2(M) + w_1(M)^2$ we have that $w_2(M) = w_1(M)^2$ precisely when all self intersections in M are even. Moreover

$$\begin{aligned} v_2(M) &= p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi)) + p^*(w_1(B) + w_1(\xi))^2 \\ &= p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi) + w_1(B)^2 + w_1(\xi)^2) \\ &= p^*(w_2(\xi)) \quad \text{from the Wu relations for } B, \end{aligned}$$

and as p^* is injective we obtain $w_2(\xi) = 0 \iff v_2(M) = 0$.

If ξ is a geometric S^2 -bundle with fundamental group $\Pi < O(3) \times \text{Isom}(\mathbb{X}^2)$, then $w_1(\xi)$ is detected by the determinant:

$$\det(g_S) = (-1)^{w_1(\xi)(g_S, g_X)} \quad \text{for all } (g_S, g_X) \in \Pi.$$

GEOMETRIC 4-MANIFOLDS THAT ARISE AS TOTAL SPACES
OF S^2 -BUNDLES OVER CLOSED SURFACES

Before proving that all S^2 -bundles over closed aspherical surfaces are geometric we shall determine which geometric 4-manifolds are total spaces of S^2 -bundles over closed aspherical surfaces.

Lemma. *Let M be a closed $S^2 \times \mathbb{X}^2$ -manifold with fundamental group $\Pi_1(M) < \text{Isom}(S^2 \times \mathbb{X}^2)$ and $\Pi_{\mathbb{X}^2}$ denote the image of $\Pi_1(M)$ in $\text{Isom}(\mathbb{X}^2)$.*

- (1) *If $\Pi_1(M)$ is torsion free, then it acts freely on $\mathbb{X}^2$ and $M = (S^2 \times \mathbb{X}^2)/\Pi_1(M)$ is the total space of an S^2 -bundle over $\mathbb{X}^2/\Pi_{\mathbb{X}^2}$.*
- (2) *$\Pi_1(M)$ has a torsion free subgroup of index two.*

Proof. (1) Let $g \in \Pi_1(M)$ fix $e \in \mathbb{X}^2$. Then $\forall s \in S^2$ the orbit of $(s, e) \in S^2 \times \mathbb{X}^2$ under $\langle g \rangle$ is homeomorphic to the orbit of $s \in S^2$ by $\langle g \rangle$; moreover the stabilizer of the orbit is trivial as $\Pi_1(M)$ acts fixed point free. Now as S^2 is compact these orbits must be finite so that $|\langle g \rangle| < \infty$ and $\Pi_1(M)$ has torsion. Similarly, the kernel of the projection of $\Pi_1(M)$ onto $\Pi_{\mathbb{X}^2}$ is a finite group which acts freely on S^2 .

In particular if $\Pi_1(M)$ is torsion free, then we have the following commutative diagram:

$$\begin{array}{ccc} S^2 \times \mathbb{X}^2 & \xrightarrow{\rho} & \mathbb{X}^2 \\ \downarrow f & & \downarrow \bar{f} \\ M = (S^2 \times \mathbb{X}^2)/\Pi_1(M) & \xrightarrow{p} & \mathbb{X}^2/\Pi_{\mathbb{X}^2} \end{array}$$

where ρ is the canonical projection and f and $\bar{f}$ are universal covering projections. We show that the fibres of p are S^2 . Let $e' = \bar{f}(e) \in \mathbb{X}^2/\Pi_{\mathbb{X}^2}$. Then $p^{-1}(e') = f\rho^{-1}\bar{f}^{-1}(e') = f\rho^{-1}(\text{Orb}_{\Pi_{\mathbb{X}^2}}(e)) = f(S^2 \times \text{Orb}_{\Pi_{\mathbb{X}^2}}(e))$. Since $\Pi_{\mathbb{X}^2}$ acts freely on $\mathbb{X}^2$, the map $S^2 \times \{e\} \mapsto f(S^2 \times \text{Orb}_{\Pi_{\mathbb{X}^2}}(e))$ is a bijection so that $p^{-1}(e') \simeq S^2$. Moreover the projection $\Pi_1(M) \twoheadrightarrow \Pi_{\mathbb{X}^2}$ is an isomorphism.

(2) Now suppose that $g = (g_S, g_X)$ is a nontrivial element of $\Pi_1(M)$ of finite order. As g_X is a self homeomorphism of the plane of finite order it fixes a point. Therefore g_S must act freely, and so $g_S = -I$ since $\{I, -I\}$ is the only nontrivial subgroup of $O(3)$ that acts freely and properly discontinuously on S^2 . Hence if we denote the projection of $\Pi_1(M)$ onto its geometric factor in $O(3)$ by α , the composite map:

$$\text{Isom}(S^2 \times \mathbb{X}^2) > \Pi_1(M) \xrightarrow{\alpha} O(3) \xrightarrow{\text{Det}} \{1, -1\} = \mathbb{Z}/2\mathbb{Z}$$

will have kernel $K = \{I\} \times \Pi_{\mathbb{X}^2} \cong \Pi_{\mathbb{X}^2} < \Pi_1(M)$ of index two and must be torsion free. $\square$

Since the fundamental groups of total spaces of S^2 -bundles over closed aspherical surfaces are torsion free, we have proved the following:

Theorem 1. *A closed $S^2 \times E^2$ - or $S^2 \times \mathbb{H}^2$ -manifold M is the total space of an S^2 -bundle over a closed aspherical surface if and only if $\Pi_1(M)$ is torsion free.*

GEOMETRIC 4-MANIFOLDS THAT ARISE AS TOTAL SPACES
OF $\mathbb{RP}^2$ -BUNDLES OVER CLOSED SURFACES

Let M be either a closed $S^2 \times E^2$ - or $S^2 \times \mathbb{H}^2$ -manifold. Suppose $\Pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times K$. By the preceding lemma we know K is torsion free and acts freely and properly discontinuously on X^2 , with compact quotient.

Let $g \in \Pi_1(M)$ generate the torsion subgroup. Viewing $\Pi_1(M)$ as a subgroup of $O(3) \times \text{Isom}(\mathbb{X}^2)$, we may write $g = (g_S, g_X)$ where $g_S \in O(3)$ and $g_X \in \text{Isom}(\mathbb{X}^2)$. Since $g_X^2 = \text{id}_{\mathbb{X}^2}$ the fixed point set $F = \text{Fix}(g_X) = \{x \in X^2 | g_X(x) = x\}$ is nonempty, and is either a point, a line, or the whole of X^2 . Since g_X commutes with the action of K on X^2 , we have $KF = F$, and so K acts freely and properly discontinuously on F . But K is neither trivial nor infinite cyclic, and so we must have $F = X^2$. Hence $g_X = \text{id}_{\mathbb{X}^2}$. It now follows as in part (2) of the above lemma that $g_S = -I$ and so $\Pi_1(M) = \langle -I \rangle \times K < O(3) \times \text{Isom}(\mathbb{X}^2)$, the product decomposition corresponding to the geometric factors. Furthermore we have the following commutative diagram:

$$\begin{array}{ccc} S^2 \times \mathbb{X} & \xrightarrow{\rho} & \mathbb{X} \\ \downarrow f & & \downarrow \bar{f} \\ M & \xrightarrow{p} & \mathbb{X}^2/K \end{array}$$

and M is the total space of an $\mathbb{RP}^2$ -bundle over $\mathbb{X}^2/K$, by an argument such as part (1) of the lemma above.

Conversely if M is the total space of an $\mathbb{RP}^2$ -bundle over B , then from the long exact homotopy sequence for a fibration we have:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} \Pi_1(M) \xrightarrow{p_*} \Pi_1(B) \longrightarrow 1 \quad .$$

Moreover $w_1(M)$ restricts nontrivially to the fibre as shown below:

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow{w_1(M)} & \mathbb{Z}/2\mathbb{Z} \\ \uparrow i & \nearrow & \\ \Pi_1(\mathbb{RP}^2) & & \end{array}$$

so that the above sequence splits and we have proved the following:

Theorem 2. *A closed $S^2 \times E^2$ - or $S^2 \times \mathbb{H}^2$ -manifold M is the total space of an $\mathbb{RP}^2$ -bundle over a closed aspherical surface if and only if $\Pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times K$, where K is torsion free.*

Now for every total space M of an $\mathbb{RP}^2$ -bundle over B , $w_1(M) \neq 0$; therefore M is nonorientable and hence has a unique orientation double cover $\widetilde{M}$. Moreover this orientation cover is the total space of an S^2 -bundle over B ; the $\mathbb{RP}^2$ -bundle being obtained from the S^2 -bundle by factoring out the fibre by the antipodal map. That is, we have a bijective correspondence:

$$\left\{ \begin{array}{c} \text{orientable } S^2\text{-bundles} \\ \text{over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{RP}^2\text{-bundles} \\ \text{over } B \end{array} \right\} ,$$

and there are two $\mathbb{RP}^2$ -bundles over each closed aspherical surface.

S^2 -BUNDLES OVER T

In the spirit of Ue we show that the nonorientable S^2 -bundles over the torus $T = S^1 \times S^1$ and the Klein bottle Kb admit geometries of type $S^2 \times E^2$ before considering the case where $\chi(B) < 0$.

Now we may take the fundamental domain of $T = \mathbb{R}^2/\mathbb{Z}^2$ to be $\mathfrak{T} = \{(s, t) \in \mathbb{R}^2 | 0 \leq s, t \leq 1\}$, and use this to calculate $w_2(\xi)$ by constructing a section $\sigma: \mathfrak{T} \rightarrow S^2 \times E^2$ of ξ so that σ induces a section $\tilde{\sigma}: T \rightarrow M$ of ξ as illustrated below:

That is, $\tilde{\sigma}(g(t)) = f\sigma(i(t)) \quad \forall t \in \mathfrak{T}$ where f and d are universal covering projections, and g is the standard gluing map. In fact the cross-sections we shall describe are maps $\sigma: \mathfrak{T} \rightarrow S^2 \times E^2$, with the action of the gluing map g predetermined by construction.

Let $R_i \in O(3)$ be the reflection in $\mathbb{R}^3$ that changes the sign of the i^{th} coordinate, for $i = 1, 2, 3$. Then from [H2] we have that the four 4-manifolds that are total spaces of S^2 -bundles over T are given by the quotient of $S^2 \times E^2$ by the subgroups of $Isom(S^2 \times E^2) = Isom(S^2) \times Isom(E^2) = O(3) \times (\mathbb{R}^2 \rtimes O(2))$ generated by $(A_j, (1, 0), I)$ and $(B_j, (0, 1), I)$ for $j = 1, 2$, each of which acts freely and cocompactly on $S^2 \times E^2$ and is isomorphic to $\mathbb{Z}^2 = \Pi_1(T) = \Pi_1(M)$, where:

- (I) $\xi_1: A_1 = I, B_1 = -I$;
- (II) $\xi_2: A_2 = R_1, B_2 = R_1 R_2$;
- (III) $\xi_3: A_3 = I, B_3 = I$; and
- (IV) $\xi_4: A_4 = R_1 R_2, B_4 = R_1 R_3$.

Cases (III) and (IV) yield orientable 4-manifolds that are total spaces of S^2 -bundles over T and have been shown to be distinguished by $w_2(\xi_3) = 0$ and $w_2(\xi_4) \neq 0$, [Ue].

Using the standard stereographic projection of $S^2 \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ onto $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we may identify the reflections $R_i: S^2 \rightarrow S^2$ with the antiholomorphic involutions:

$$z \xrightarrow{R_1} \frac{1}{\bar{z}}, \quad z \xrightarrow{R_2} \bar{z}, \quad z \xrightarrow{R_3} -\bar{z}.$$

Then

$$\sigma_\lambda(s, t) = (2t - 1)e^{i\pi(2s+\lambda)} + \lambda t(t - 1)$$

defines a 1-parameter family of cross-sections on $\mathfrak{T}$ for ξ_1 with $\sigma_0(s, t) = \sigma_1(s, t)$ only when $(s, t) \in \{(0, \frac{5-\sqrt{17}}{2}), (\frac{1}{2}, \frac{-3+\sqrt{17}}{2})\}$. The tangent plane to $\sigma_0(s, t)$ at $(0, \frac{5-\sqrt{17}}{2})$ is spanned by the 4-vectors $\{(0, 2\pi(4 - \sqrt{17}), 1, 0), (2, 0, 0, 1)\}$, and the tangent plane to $\sigma_1(s, t)$ at $(0, \frac{5-\sqrt{17}}{2})$ is spanned by the 4-vectors $\{(0, 2\pi(\sqrt{17} - 4), 1, 0), (2 - \sqrt{17}, 0, 0, 1)\}$. These two sets together constitute a linearly independent set and hence the intersection at this point is transverse. Similarly the intersection at $(\frac{1}{2}, \frac{-3+\sqrt{17}}{2})$ may be shown to be transverse so that the mod 2 intersection number

is $\sigma \cdot \sigma = 0$ [as $\sigma_0(s, t)$ and $\sigma_1(s, t)$ are clearly isotopic]. Hence the second Wu class $v_2(M) = 0$ and from above we get $w_2(\xi_1) = 0$.

Similarly $\sigma_\lambda(s, t) = (\lambda s(s-1) + 1)^{(t^2+t-1)} e^{i\pi(2t+\lambda)(2t-1)}$ defines a 1-parameter family of cross-sections on $\mathfrak{T}$ for ξ_2 such that the sections σ_0 and σ_1 intersect transversely in a single point. Hence the intersection number $\sigma \cdot \sigma = 1$, thus $v_2(M) \neq 0$ and $w_2(\xi_2) \neq 0$.

Thus these two bundles are distinguished by their second Stiefel-Whitney classes so that all S^2 -bundles over T admit geometries of type $S^2 \times E^2$.

S^2 -BUNDLES OVER Kb

The following six pairs $((A_j, (1, 0), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}), (B_j, (0, 1), I))$ for $1 \leq j \leq 6$ of elements of $Isom(S^2 \times E^2)$ each generate a subgroup isomorphic to $\mathbb{Z} \tilde{\times} \mathbb{Z} = \Pi_1(Kb) = \Pi_1(M)$ which acts freely and cocompactly on $S^2 \times E^2$, where:

- (I) $\xi_1 : A_1 = I, B_1 = I$;
- (II) $\xi_2 : A_2 = I, B_2 = -I$;
- (III) $\xi_3 : A_3 = I, B_3 = R_2 R_3$;
- (IV) $\xi_4 : A_4 = R_1 R_2, B_4 = R_1$;
- (V) $\xi_5 : A_5 = -I, B_5 = I$; and
- (VI) $\xi_6 : A_6 = R_1, B_6 = R_2 R_3$.

Remark. The generators given on page 30 of [H1] in the cases (viii) and (ix) are incorrect. The pair (A, B) should be $(I, R_2 R_3)$ or $(R_1, R_2 R_3)$ respectively.

Cases (V) and (VI) yield orientable 4-manifolds that are total spaces of S^2 -bundles over Kb and have been shown to be distinguished by $w_2(\xi_5) = 0$ and $w_2(\xi_6) \neq 0$, [Ue].

Note that in cases (I)-(IV) the generator $(A_j, (1, 0), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ is an orientation reversing homeomorphism of $S^2 \times E^2$ so that the corresponding 4-manifolds are nonorientable. These 4-manifolds are all total spaces of S^2 -bundles over Kb , and we shall show that they may be distinguished by the Stiefel-Whitney classes of these bundles.

Since ξ_1 is the product bundle, $w_2(\xi_1) = 0$ and $w_1(\xi_1) = 0$.

Let the fundamental domain of $Kb = \mathbb{R}^2 / (\mathbb{Z} \tilde{\times} \mathbb{Z})$ be $\mathfrak{K} = \{(s, t) \in \mathbb{R}^2 | 0 \leq s \leq 1, -\frac{1}{2} \leq t \leq \frac{1}{2}\}$. The functions $\sigma_\lambda(s, t) = (t^2 + \frac{3}{4})e^{i\pi(2s+\lambda)} + \lambda(t^2 - \frac{1}{4})$ define a 1-parameter family of cross-sections on $\mathfrak{K}$ for ξ_2 that induces a cross-section $\tilde{\sigma} : Kb \rightarrow M$ of Kb as above with the sections σ_0 and σ_1 disjoint, so that $\sigma \cdot \sigma = 0$. That is, $v_2(M) = 0$ and therefore $w_2(\xi_2) = 0$. Also the orientation character $w_1(\xi_2)$ is nonzero as $\det(-I) = -1$.

The functions $\sigma_\lambda(s, t) = \lambda t e^{i\pi s} + (1 - \lambda)(t^2 - \frac{1}{4})e^{i\pi 2s}$ define a 1-parameter family of cross-sections for ξ_3 such that the sections σ_0 and σ_1 intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and therefore $w_2(\xi_3) \neq 0$. Furthermore $w_1(\xi_3) = 0$ as $\det(I) = \det(R_2 R_3) = 1$.

For ξ_4 the functions $\sigma_\lambda(s, t) = (\lambda(t^2 - \frac{1}{4}) + 1)^{(s^2+s-1)} e^{i\pi(2s+\lambda)(2s-1)}$ define a 1-parameter family of cross-sections with the sections σ_0 and $\sigma_1(s, t)$ intersecting transversely in a point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and therefore $w_2(\xi_4) \neq 0$. Also $w_1(\xi_4) \neq 0$ as $\det(R_1) = -1$.

Thus all S^2 -bundles over Kb are geometric of type $S^2 \times E^2$.

S^2 -BUNDLES OVER CLOSED ORIENTABLE SURFACES WITH $\chi < 0$

Let T^g be the closed orientable surface of genus g . Then

$$\Pi_1(T^g) = \langle X_1, Y_1, \dots, X_g, Y_g \mid \prod_{i=1}^g [X_i, Y_i] = 1 \rangle$$

and we have a degree 1 map $\Omega: \Pi_1(T^g) \longrightarrow \mathbb{Z}^2$ that kills the generators X_j, Y_j for $j > 1$. If $\mathfrak{T}^g \subset \mathbb{H}^2$ is a $2g$ -gon representing the fundamental domain of T^g , then Ω may be induced by a ‘collapsing function’ $\Omega: \mathfrak{T}^g \longrightarrow \mathfrak{T}$ that collapses $2g - 4$ sides of $\mathfrak{T}^g$ to a single vertex in the rectangle $\mathfrak{T}$. In turn this induces a degree 1 collapsing function $\hat{\Omega}: T^g \longrightarrow T$ that collapses $g - 1$ handles on T^g to a single point on T . Hence given an S^2 -bundle $\xi, p: M_\alpha \rightarrow T$ over T where $M_\alpha = (S^2 \times E^2)/\Gamma_\alpha$ and $\Gamma_\alpha = \{(\alpha(h), h) \mid h \in \Pi_1(T)\} \leq \text{Isom}(S^2 \times E^2)$ where $\alpha: \mathbb{Z}^2 \rightarrow O(3)$ is as before [there are four choices for α]; we may pull back along $\hat{\Omega}$ to get the commutative diagram:

$$\begin{array}{ccc} M_{\alpha\Omega} & \longrightarrow & M_\alpha \\ \downarrow p_{\alpha\Omega} & & \downarrow p \\ T^g & \xrightarrow{\hat{\Omega}} & T \end{array}$$

and an S^2 -bundle $\hat{\Omega}^*(\xi), p_{\alpha\Omega}: M_{\alpha\Omega} \rightarrow T^g$ over T^g , where $M_{\alpha\Omega} = (S^2 \times \mathbb{H}^2)/\Gamma_{\alpha\Omega}$ and $\Gamma_{\alpha\Omega} = \{(\alpha\Omega(h), h) \mid h \in \Pi_1(T^g)\} \leq \text{Isom}(S^2 \times \mathbb{H}^2)$. As $\hat{\Omega}$ is of degree 1, it induces monomorphisms in cohomology so that all the Stiefel-Whitney classes are carried from ξ to $\hat{\Omega}^*(\xi)$, that is, $w(\hat{\Omega}^*(\xi)) = \hat{\Omega}^*(\xi) = w(\xi)$. Hence all S^2 -bundles over T^g for $g \geq 2$ are geometric of type $S^2 \times \mathbb{H}^2$.

S^2 -BUNDLES OVER CLOSED NONORIENTABLE SURFACES WITH $\chi < 0$

Let B be the closed surface $\#^3\mathbb{RP}^2 = T \# \mathbb{RP}^2 = Kb \# \mathbb{RP}^2$. Then we have a collapsing function map $\hat{\Omega}: T \# \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ that collapses the torus summand to a single point. This map $\hat{\Omega}$ is of degree 1 and so induces monomorphisms in cohomology. In particular $\hat{\Omega}^*$ preserves the orientation character, that is, $w_1(\hat{\Omega}^*(\xi)) = \hat{\Omega}^*w_1(\mathbb{RP}^2) = w_1(B)$, and is an isomorphism on H^2 . We may pull back the four S^2 -bundles ξ over $\mathbb{RP}^2$ along $\hat{\Omega}$ to obtain the four bundles over B with first Stiefel-Whitney class $w_1(\hat{\Omega}^*\xi)$ either 0 or $w_1(B)$.

Similarly we have a collapsing function $\hat{\Gamma}: Kb \# \mathbb{RP}^2 \longrightarrow \mathbb{RP}^2$ that collapses the Klein bottle summand to a single point. This map $\hat{\Gamma}$ is of degree 1 mod 2 so that $\hat{\Gamma}^*w_1(\mathbb{RP}^2)$ has nonzero square since $w_1(\mathbb{RP}^2)^2 \neq 0$. Note that in this case $\hat{\Gamma}^*w_1(\mathbb{RP}^2) \neq w_1(B)$. Hence we may pull back the two S^2 -bundles ξ over $\mathbb{RP}^2$ with $w_1(\xi) = w_1(\mathbb{RP}^2)$ to obtain a further two bundles over B with $w_1(\hat{\Gamma}^*(\xi))^2 = \hat{\Gamma}^*w_1(\xi)^2 \neq 0$, as $\hat{\Gamma}$ is a ring monomorphism.

Again there is a collapsing function $\hat{\Delta}: Kb \# \mathbb{RP}^2 \longrightarrow Kb$ that collapses the Klein bottle summand to a single point. Once again $\hat{\Delta}$ is of degree 1 mod 2 so that we may pull back the two S^2 -bundles ξ over Kb with $w_1(\xi) = w_1(Kb)$ along $\hat{\Delta}$ to obtain the remaining two S^2 -bundles over B . These two bundles $\hat{\Delta}^*(\xi)$ have $w_1(\hat{\Delta}^*(\xi)) \neq 0$ but $w_1(\hat{\Delta}^*(\xi))^2 = 0$, as $w_1(Kb) \neq 0$ but $w_1(Kb)^2 = 0$ and $\hat{\Delta}^*$ is a monomorphism.

Similar arguments apply to bundles over $\#^n \mathbb{RP}^2$ where $n > 3$.

Thus all S^2 -bundles over all closed aspherical surfaces are geometric. Furthermore since the antipodal involution of a geometric S^2 -bundle is induced by an isometry $(-I, \text{id}_{\mathbb{X}^2}) \in O(3) \times \text{Isom}(\mathbb{X}^2)$, we have that all $\mathbb{RP}^2$ -bundles over closed aspherical surfaces are geometric. These final two results together with Theorems 1 and 2 prove the main theorem.

An alternative route to the main theorem would be to first show that orientable 4-manifolds which are total spaces of S^2 -bundles are geometric, then deduce that $\mathbb{RP}^2$ -bundles are geometric [as above], and finally observe that every S^2 -bundle space double covers an $\mathbb{RP}^2$ -bundle space.

REFERENCES

- [H1] Jonathan A. Hillman, *On 4-manifolds with universal covering space $S^2 \times \mathbb{R}^2$ or $S^3 \times \mathbb{R}$* , Top. Appl. **52** (1993), 23-42. MR **95b**:57020
- [H2] Jonathan A. Hillman, *On 4-manifolds with universal covering space a compact geometric manifold*, J. Austral. Math. Soc. (Series A). **55** (1993), 137-148. MR **94i**:57031
- [Hu] Dale Husemoller, *Fibre bundles*, Springer-Verlag, New York, 1994. MR **94k**:55001
- [Me] Paul Melvin, *2-sphere bundles over compact surfaces*, Proc. Amer. Math. Soc. **92** (1984), 567-572. MR **85j**:57039
- [Ue] Masaaki Ue, *Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II*, J. Math. Soc. Japan. **43** (1991), 149-183. MR **91m**:57019

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