A MONOTONEITY PROPERTY OF THE GAMMA FUNCTION

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ABSTRACT. In this paper we obtain a monotoneity property for the gamma function that yields sharp asymptotic estimates for $\Gamma(x)$ as x tends to ∞ , thus proving a conjecture about $\Gamma(x)$.

1. INTRODUCTION

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative Ψ , the so-called digamma function, are defined as

$$(1.1) \quad \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For extensions of these functions to complex variables and for basic properties see [WW].

Over the past half century many authors have obtained inequalities for these important functions (see [A1], [A2] and bibliographies in those papers). In keeping with this tradition we here obtain a monotoneity property of the gamma function that yields a sharp asymptotic estimate for $\Gamma(x)$ as x tends to ∞ .

In [AVV, Lemma 2.39] the following result was obtained.

1.2. Lemma.

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{\log \Gamma(1 + \frac{x}{2})}{x \log x} = \frac{1}{2},$$

and

$$(1.4) \quad f(x) \equiv \frac{1}{x} \log \Gamma(1 + \frac{x}{2})$$

is strictly increasing from $[2, \infty]$ onto $[0, \infty)$.

It was conjectured in [AVV, Remark 2.41] that the function in (1.3) is strictly increasing from $[2, \infty)$ onto $[0, 1/2)$. In order to obtain an affirmative answer to the above conjecture we prove here the following result.

1.5. Theorem. *The function $f(x) \equiv (\log \Gamma(x+1))/(x \log x)$ is strictly increasing from $(1, \infty)$ onto $(1-\gamma, 1)$, where γ is the Euler-Mascheroni constant. In particular, for $x \in (1, \infty)$,*

$$(1.6) \quad x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}$$

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and

$$(1.7) \quad \lim_{x \rightarrow \infty} \frac{\log \Gamma(x)}{(x-1) \log(x-1)} = 1.$$

In [AVV] the the following result was also obtained.

1.8. Lemma. *Let $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ denote the n -dimensional volume of the unit ball B^n in $\mathbb{R}^n$. Then*

$$(1.9) \quad \lim_{n \rightarrow \infty} \Omega_n^{1/(n \log n)} = e^{-\frac{1}{2}},$$

$$(1.10) \quad \Omega_n^{1/n} \text{ decreases strictly to } 0 \text{ as } n \rightarrow \infty,$$

$$(1.11) \quad \sum_{n=2}^{\infty} \Omega_n^{1/\log n} \text{ is convergent.}$$

It was pointed out in [AVV, Remark 2.41] that if the function in (1.3) above has the conjectured property this would imply that $\Omega_n^{1/(n \log n)}$ is strictly decreasing for $n \geq 2$. Thus our Theorem 1.5 implies this monotonicity of $\Omega_n^{1/(n \log n)}$ (see Corollary 3.1). It should be observed that Ω_n itself is not monotone [BH, pp. 263, 264] (cf. [SV]).

In this paper we let $\mathbb{N}$ denote the set of positive integers and, for the real number x , let $[x]$ denote the integer satisfying $x-1 < [x] \leq x$.

2. PRELIMINARY RESULTS

Before establishing the main theorem we need to prove some technical lemmas.

2.1. Lemma. *The function $f(x) \equiv \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3}$ is positive for $x \in [1, 4)$.*

Proof. Let $u(t, x) = (t-x)/(t+x)^3$, for $t, x \in [1, \infty)$. Then u is strictly decreasing in t on $[2x, \infty)$ for any $x \in [1, \infty)$, since $\partial u / \partial t = 2(2x-t)/(t+x)^4$.

For $k \in \mathbb{N}$ and $x \geq 1$, let $v_k = u(k, x)$. Then $v_k(x)$ is strictly decreasing in k for $k \in \mathbb{N} \cap [2x, \infty)$ and

$$v_k(x) = u(k, x) > u(t, x) \quad \text{for } t > k \geq 2x.$$

Hence, for $k \geq 2x \geq 2$,

$$v_k(x) = \frac{k-x}{(k+x)^3} = \int_k^{k+1} v_k(x) dt > \int_k^{k+1} u(t, x) dt,$$

so that

$$(2.2) \quad \begin{cases} \sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=2[x]+2}^{\infty} \int_n^{n+1} u(t, x) dt = \int_{2[x]+x}^{\infty} \frac{t-x}{(t+x)^3} dt \\ \quad = \int_{2[x]+2}^{\infty} \left[\frac{1}{(t+x)^2} - \frac{2x}{(t+x)^3} \right] dt = \frac{2(1+[x])}{(2[x]+x+2)^2}. \end{cases}$$

It follows from (2.2) that

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{2[x]+1} v_n(x) + \sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=1}^{2[x]+1} v_n(x) + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
 &= \left[\frac{1-x}{(1+x)^3} + \frac{2-x}{(2+x)^3} + \cdots + \frac{[x]-x}{([x]+x)^3} \right] \\
 &\quad + \left[\frac{[x]+1-x}{([x]+x+1)^3} + \frac{[x]+2-x}{([x]+x+2)^3} + \cdots + \frac{2[x]+1-x}{(2[x]+x+1)^3} \right] \\
 &\quad + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
 &\geq \frac{1}{(x+1)^3} [(1+x) + (2-x) + \cdots + ([x]-x)] \\
 &\quad + \frac{1}{(2[x]+x+1)^3} [([x]+1-x) + ([x]+2-x) + \cdots + (2[x]+1-x)] \\
 &\quad + 2 \frac{[x]+1}{(2[x]+x+2)^2} \\
 &= \frac{[x]([x]+1-2x)}{2(x+1)^3} + \frac{([x]+1)(3[x]+2-2x)}{2(2[x]+x+1)^3} + \frac{2[x]+1}{(2[x]+x+2)^2} \\
 &\equiv f_1(x).
 \end{aligned}$$

If $1 \leq x < 2$, then $[x] = 1$ and

$$(2.3) \quad \begin{cases} f_1(x) = \frac{1-x}{(x+1)^3} + \frac{5-2x}{(3+x)^3} + \frac{4}{(4+x)^2} \\ \quad = \frac{1}{(x+1)^3(4+x)^2} [3x^3 + 5x^2 + 4x + 20] + \frac{5-2x}{(3+x)^3} > 0. \end{cases}$$

If $2 \leq x < 3$, then $[x] = 2$ and

$$(2.4) \quad \begin{cases} f_1(x) = \frac{3-2x}{(x+1)^3} + 3 \frac{4-x}{(5+x)^3} + \frac{6}{(6+x)^2} \\ \quad = \frac{4x^3 - 3x^2 - 18x + 114}{(x+1)^3(6+x)^2} + 3 \frac{4-x}{(5+x)^3} \\ \quad > \frac{4x^3 + 33}{(x+1)^3(6+x)^2} + 3 \frac{4-x}{(5+x)^3} > 0. \end{cases}$$

If $3 \leq x < 4$, then $[x] = 3$ and

$$(2.5) \quad \left\{ \begin{aligned} f_1(x) &= \frac{3(2-x)}{(x+1)^3} + \frac{2(11-2x)}{(7+x)^3} + \frac{8}{(8+x)^2} \\ &= \frac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3(8+x)^2} + 2\frac{11-2x}{(7+x)^3} \\ &> \frac{1}{(8+x)^2} \left[\frac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3} + \frac{22-4x}{7+x} \right] \\ &= \frac{x^4 + 27x^3 - 144x^2 - 50x + 2766}{(x+8)^2(x+7)(x+1)^3} \\ &> \frac{1072}{(x+8)^2(x+7)(x+1)^3} > 0. \end{aligned} \right.$$

The conclusion now follows from (2.2) – (2.5). $\square$

2.6. Lemma. *The function $g(x) \equiv x^2\Psi'(1+x) - x\Psi(1+x) + \log\Gamma(1+x)$ is positive for all $x \in [1, \infty)$.*

Proof. From the well-known difference equation $\Gamma(x+1) = x\Gamma(x)$ [WW, p. 237] it follows easily that

$$(2.7) \quad \Psi(x+1) = \frac{1}{x} + \Psi(x),$$

from which we obtain

$$(2.8) \quad xg'(x) = x^2\Psi'(x) + x^3\Psi''(x) + 1 \equiv g_1(x).$$

Since

$$\Psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}, \quad \Psi''(x) = -2 \sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$$

[Ah, p. 200, (31)], g_1 can be rewritten as

$$(2.9) \quad g_1(x) = x^2 \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3} = x^2 f(x),$$

where f is as in Lemma 2.1.

From Lemma 2.1, (2.8), and (2.9), we see that g is strictly increasing on $[1, 4]$, and hence, by [W, Exercise 2, p. 80] and [Ah, p. 199, (29)],

$$(2.10) \quad g(x) \geq g(1) = \Psi'(2) - \Psi(2) = \sum_{n=2}^{\infty} \frac{1}{n^2} - 1 + \gamma = \frac{\pi^2}{6} + \gamma - 2 = 0.2221 \dots > 0$$

for $x \in [1, 4]$.

Next, since

$$(2.11) \quad \left\{ \begin{aligned} \frac{1}{x} &< \Psi'(x) < \frac{1}{x-1}, \\ \log x - \frac{1}{x} &< \Psi(x) < \log x - \frac{1}{2x}, \end{aligned} \right.$$

for $x > 1$ (see [ABRVV, Theorem 3.1], [S, Lemma 4b], [A2, (2.2)]), it follows from (2.7) that

$$(2.12) \quad \begin{cases} g(x) = x^2 \Psi'(x) - x \Psi(x) + \log x + \log \Gamma(x) - 2 \\ > x + (1-x) \log x + \log \Gamma(x) - \frac{3}{2} \equiv g_2(x). \end{cases}$$

Differentiation gives

$$g_2'(x) = \Psi(x) - \left(\log x - \frac{1}{x}\right), \quad x > 1,$$

which is positive by (2.11). Hence g_2 is strictly increasing on $[1, \infty)$ so that, for $x \in [4, \infty)$,

$$(2.13) \quad g(x) > g_2(x) \geq g_2(4) = \frac{5}{2} - 3 \log 4 + \log 6 = 0.1328 \dots > 0.$$

The result now follows from (2.10) and (2.13). $\square$

2.14. Lemma. *The function $h(x) \equiv x \Psi(1+x) - \log \Gamma(1+x)$ is strictly increasing from $[0, \infty)$ onto $[0, \infty)$. Moreover,*

$$(2.15) \quad \lim_{x \rightarrow \infty} \frac{h(x)}{x} = 1 \quad \text{and} \quad \frac{h(x)}{x} = 1 + O\left(\frac{\log x}{x}\right)$$

as $x \rightarrow \infty$.

Proof. Differentiation gives

$$h'(x) = x \Psi'(x+1) > 0,$$

and the monotonicity of h follows. Clearly, $h(0) = 0$. Since

$$(2.16) \quad \begin{cases} \log \Gamma(x) = (x - \frac{1}{2}) \log x - x + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{x}\right), \\ \Psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \end{cases}$$

as $x \rightarrow \infty$ by [S, Theorems 4, 5], we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{h(x)}{x} &= \lim_{x \rightarrow \infty} \left[\log(1+x) - \frac{1}{2(x+1)} - \frac{1}{x} \left(x + \frac{1}{2}\right) \log(x+1) + \frac{x+1}{x} \right] \\ &= \lim_{x \rightarrow \infty} \left[1 - \frac{\log(1+x)}{2x} \right] = 1. \end{aligned}$$

$\square$

2.17. Lemma. *The function $H(x) \equiv \log x - \frac{1}{h(x)} \log \Gamma(x+1)$ is strictly increasing from $[1, \infty)$ onto $[0, 1)$. Here h is as in Lemma 2.14. In particular, for all $x \in (1, \infty)$,*

$$(2.18) \quad 1 - \frac{1}{\log x} < \frac{\log \Gamma(x+1)}{x \Psi(x+1)} < 1 - \frac{1}{1 + \log x}.$$

Proof. Clearly, $H(1) = 0$. It follows from (2.15) and (2.16) that

$$\begin{aligned}\lim_{x \rightarrow \infty} H(x) &= \lim_{x \rightarrow \infty} \left[\left(1 - \frac{x}{h(x)}\right) \log x + \frac{x}{h(x)} \left(-\frac{\log x}{2x} + 1 - \frac{\log(2\pi)}{2x} + O\left(\frac{1}{x^2}\right)\right) \right] \\ &= \lim_{x \rightarrow \infty} \left[O\left(\frac{(\log x)^2}{h(x)}\right) + \frac{x}{h(x)} + O\left(\frac{1}{x}\right) \right] \\ &= 1 + \lim_{x \rightarrow \infty} O\left(\frac{1}{x}(\log x)^2\right) = 1.\end{aligned}$$

Next, by differentiation, we get

$$\begin{aligned}H'(x) &= \frac{1}{x} - \frac{1}{(h(x))^2} [\Psi(x+1)h(x) - x\Psi'(x+1)\log \Gamma(x+1)] \\ &= \frac{1}{x} - \frac{1}{x(h(x))^2} [(x\Psi(x+1) - \log \Gamma(x+1) + \log \Gamma(x+1))h(x) \\ &\quad - x^2\Psi'(x+1)\log \Gamma(x+1)] \\ &= \frac{\log \Gamma(x+1)}{x(h(x))^2} [x^2\Psi'(x+1) - h(x)] \\ &= \frac{g(x)}{x(h(x))^2} \log \Gamma(x+1),\end{aligned}$$

where g is as in Lemma 2.6. Hence the monotoneity of H follows from Lemma 2.6. The inequality (2.18) is clear. $\square$

3. PROOF OF THE MAIN THEOREM

We now show how Theorem 1.5 follows from the lemmas in Section 2. By differentiation we get

$$(x \log x)^2 f'(x) = h(x)H(x),$$

where $f(x) = (\log \Gamma(x+1))/(x \log x)$ and where h and H are as in Lemmas 2.14 and 2.17, respectively. Hence, the monotoneity of f follows from Lemmas 2.14 and 2.17.

Next, by l'Hôpital's Rule, we have

$$f(1^+) = \lim_{x \rightarrow 1} \frac{\Psi(x+1)}{1 + \log x} = \Psi(2) = -1 - \gamma$$

[AS, 6.3.3] and, by (2.16),

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{(x + \frac{1}{2}) \log(x+1) - (x+1) + \frac{1}{2} \log(2\pi)}{x \log x} \\ &= \lim_{x \rightarrow \infty} \frac{\log(x+1)}{\log x} = 1.\end{aligned}$$

Inequality (1.6) and limit (1.7) are clear. $\square$

3.1. Corollary. (1) The function $f(x) \equiv (\log \Gamma(1 + \frac{x}{2})) / (x \log x)$ is strictly increasing from $[2, \infty)$ onto $[0, 1/2)$ (the conjecture in [AVV, Remark 2.41] is true).

(2) For $n \in \mathbb{N}$, let $\Omega_n = \pi^{n/2} / \Gamma(1 + n/2)$ be the n -dimensional volume of the unit ball B^n in $\mathbb{R}^n$. Then the sequence $G(n) \equiv \Omega_n^{1/(n \log n)}$ is strictly decreasing for $n \geq 2$, with $G(2) = \pi^{1/\log 4}$ and $\lim_{n \rightarrow \infty} G(n) = e^{-1/2}$.

Proof. (1) Let $t = x/2$. Then

$$f(x) = \frac{\log t}{2 \log(2t)} \cdot \frac{\log \Gamma(t+1)}{t \log t},$$

and the conclusion follows from Theorem 1.5 since the function $(\log t) / \log(2t)$ is strictly increasing from $[1, \infty)$ onto $[0, 1)$.

(2) Since

$$\log G(n) = \frac{1}{2} \frac{\log \pi}{\log n} - \frac{\log \Gamma(1 + \frac{n}{2})}{n \log n},$$

the assertion follows from part (1). $\square$

3.2. Remark. By methods similar to those used to prove Theorem 1.5 we can show that the function $f(x) \equiv (\log \Gamma(x+1)) / ((x-1) \log(2x))$ is strictly increasing from $[4.5, \infty)$ onto $[c, 1)$, where $c = (\log(135\sqrt{\pi}/32)) / (7 \log 3)$. However, f is not monotone on $(1, \infty)$ since $f'(x) < 0$ when x is near 1.

3.3. Conjecture. $f(x) \equiv (\log \Gamma(x+1)) / (x \log x)$ is concave on $(1, \infty)$.

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