FACTORIZATION OF AN INTEGRALLY CLOSED IDEAL IN TWO-DIMENSIONAL REGULAR LOCAL RINGS

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ABSTRACT. Let (R, m, k) be a two-dimensional regular local ring with algebraically closed residue field k and I be an m -primary integrally closed ideal in R . Let $T(I)$ be the set of Rees valuations of I and $k(v)$ be the residue field of the valuation ring V associated with $v \in T(I)$. Assume that (a, b) is any minimal reduction of I . We show that if I is the product of the distinct simple m -primary integrally closed ideals in (R, m, k) , then $k(v)$ is generated by the image of a/b over k for all $v \in T(I)$, and the converse of this is also true.

1. INTRODUCTION

Throughout this paper (R, m, k) will denote a 2-dimensional regular local ring (**RLR** for short) with residue field k and quotient field K . Let I be an m -primary integrally closed ideal in (R, m, k) . Concerning the structure of the integrally closed ideals in a 2-dimensional **RLR** (R, m, k) , O. Zariski proved three beautiful theorems which are the main background for this paper. Zariski's Product Theorem ([8], Appendix 5, Theorem 2') says that any product of integrally closed ideals is integrally closed. Hence the set of Rees valuations of I is

$$T(I) = \bigcup_{\substack{a \neq 0 \\ a \in I}} \left\{ v \mid v \text{ is the valuation of } (R[I/a])_q, q \in \text{Min}(aR[I/a]) \right\}.$$

By Zariski's Unique Factorization Theorem ([8], Appendix 5, Theorem 3), $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$, $\mu_i \geq 1$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in (R, m, k) . Zariski also set up an one-to-one correspondence between the set of simple integrally closed ideals of R and the set of prime divisors of the second kind on R ([8], Appendix 5, Theorem E). Therefore, if $\text{Min}(mR[I/a]) = \{q_1, \dots, q_\lambda\}$ then $\lambda = l$ and, upon reordering, v_i is the valuation of $(R[I/a])_{q_i}$, where v_i is the prime divisor associated to the ideal I_i for $i = 1, \dots, l$.

We denote by $k(v)$ the residue field of the valuation ring V associated with v . Assume that k is an algebraically closed field. Then $k(v)$ is a simple transcendental field extension of k for all $v \in T(I)$. Moreover, if (a, b) is any minimal reduction of I , then for all $v \in T(I)$, the image of a/b in $k(v)$ is transcendental over k . We show that if $I = I_1 \cdots I_l$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in (R, m, k) , then $k(v)$ is generated by the image of a/b over k for

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all $v \in T(I)$, and the converse of this is also true. Section 2 is devoted to some preliminaries. In section 3, we will prove main results.

2. PRELIMINARIES

Let (A, n) be a local ring and I an ideal of A . An ideal J contained in I is called a reduction of I if $J I^s = I^{s+1}$ for some integer $s \geq 0$. A reduction J of I is called a minimal reduction of I if J is minimal with respect to being a reduction of I . The order $o(I)$ of an ideal I of a local ring (A, n) is r if $I \subseteq n^r$ but $I \not\subseteq n^{r+1}$. We will use notation $e(I)$ to denote the multiplicity of an n -primary ideal I of A . Recall that an ideal is simple if it is not the unit ideal and has no nontrivial factorization. An element $a \in A$ is said to be integral over an ideal I of A if a satisfies an equation of the form

$$a^n + r_1 a^{n-1} + \cdots + r_n = 0, \quad r_i \in I^i.$$

The set of all elements in A which are integral over an ideal I forms an ideal, denoted by $\bar{I}$ and called the integral closure of I . An ideal I is said to be integrally closed (or equivalently “complete”) if $I = \bar{I}$.

In a d -dimensional local domain (A, n, l) with quotient field L , by a prime divisor of the second kind on A (or equivalently prime divisor of (A, n)) we mean a discrete valuation v of L on A which is non-negative on A and has center n on A and whose residual transcendence degree (denoted by $\text{tr.deg}_k(v)$) is $d - 1$.

Lemma 2.1. *Let (A, n) be a quasilocal normal domain with quotient field L . If $u \in L \setminus A$ is such that $u^{-1} \notin A$, then $nA[u]$ is a prime ideal in $A[u]$ and $A[u]/nA[u] \cong (A/n)[X]$, a polynomial ring in one-variable over the field A/n .*

Proof. Define the canonical homomorphism ϕ from $A[X]$ onto $A[u]$ with $\phi(X) = u$. $\text{Ker}(\phi)$ is a prime ideal in $A[X]$ since $A[u]$ is a domain. By Theorem 11.13. in [7], $\text{Ker}(\phi)$ is generated by linear polynomials $cX - d$ with $u = d/c$, where $c, d \in A$. It is not difficult to see that c and d are in the maximal ideal n of A . Hence we have the following exact sequence :

$$0 \longrightarrow \frac{nA[X]}{\text{Ker}(\phi)} \longrightarrow \frac{A[X]}{\text{Ker}(\phi)} \xrightarrow{\bar{\phi}} \frac{A[u]}{nA[u]} \longrightarrow 0,$$

where $\bar{\phi}$ is the map induced by ϕ . Hence we have $A[u]/nA[u] \cong A[X]/nA[X]$ is a polynomial ring one-variable over the field A/n , and so $nA[u]$ is a prime ideal in $A[u]$. $\square$

3. MAIN RESULTS

The following lemma will play a key role in the proofs of the first main result.

Lemma 3.1. *Let I be an m -primary integrally closed ideal in (R, m, k) . Assume that $\text{Min}(mR[It]) = \{p_1, \dots, p_l\}$ and that k is an algebraically closed field. Let v_i be the valuation of $R[It]_{p_i} \cap K$ for $i = 1, \dots, l$. Then the following conditions are equivalent.*

- (1) $R[It]/p_i$ is regular for $i = 1, \dots, l$.
- (2) $I = I_1 \cdots I_l$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in R .

- (3) For any reduction (a, b) of I , there exist elements $c_{i_1}, \dots, c_{i_n}$ in I such that $a, b, c_{i_1}, \dots, c_{i_n}$ is a minimal generating set of I and $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $i = 1, \dots, l$ and $j = 1, \dots, n$.
- (4) There exist a reduction (a, b) of I and elements $c_{i_1}, \dots, c_{i_n}$ in I such that $a, b, c_{i_1}, \dots, c_{i_n}$ is a minimal generating set of I and $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $i = 1, \dots, l$ and $j = 1, \dots, n$.

Proof. (1) $\iff$ (2) See ([5], Theorem 3.1).

(1) $\iff$ (3) $\iff$ (4) See ([5], Theorem 3.3). $\square$

Theorem 3.2. Let I be an m -primary integrally closed ideal in (R, m, k) with an algebraically closed field k and (a, b) be any minimal reduction of I . Assume that $I = I_1 \cdots I_l$, where $I_1, \dots, I_l$ are distinct m -primary simple integrally closed ideals in R . Then $k(v)$ is generated by the image of a/b over k for all $v \in T(I)$.

Proof. Suppose that $I = I_1 \cdots I_l$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in R . Let v_i be a prime divisor of R associated to I_i for $i = 1, \dots, l$. Let $\text{Min}(mR[I/b]) = \{q_1, \dots, q_\lambda\}$. By Zariski's One-to-One Correspondence Theorem, we have that $\lambda = l$ and, upon reordering, v_i is the discrete valuation of $R[I/b]_{q_i}$ for $i = 1, \dots, l$, i.e., $T(I) = \{v_1, \dots, v_l\}$. By Lemma 3.1, there exist elements $c_{i_1}, \dots, c_{i_n}$ in I such that $a, b, c_{i_1}, \dots, c_{i_n}$ is a minimal generating set of I and $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $i = 1, \dots, l$ and $j = 1, \dots, n$. For each $i = 1, \dots, l$, let $J_i = (m, c_{i_1}/b, \dots, c_{i_n}/b)$. Then $J_i \subseteq q_i$ since $v_i(c_{i_j}) > v_i(a) = v_i(b)$ for $j = 1, \dots, n$. And we have

$$\begin{aligned} R[I/b]/J_i &= R[a/b]/mR[a/b] \\ &\cong (R/m)[X] \quad \text{by Lemma 3.1.} \end{aligned}$$

Let $(a/b)_i^*$ be the image of a/b in $R[I/b]/J_i$ for $i = 1, \dots, l$. Since J_i is a prime ideal in $R[I/b]$ and $\dim(R[I/b]/J_i) = \dim(R[I/b]/q_i) = 1$, we have $J_i = q_i$ for $i = 1, \dots, l$. Thus $R[I/b]/q_i = (R/m)[(a/b)_i^*]$ for $i = 1, \dots, l$, which is a polynomial ring in one-variable over k . Localizing at q_i , we have $k(v_i) = (R/m)((a/b)_i^*)$ for $i = 1, \dots, l$. $\square$

Corollary 3.3 ([3], Remark 3.5). Let I be an m -primary integrally closed ideal in (R, m, k) with an algebraically closed field k and (a, b) be any minimal reduction of I . Assume that I is simple. Let v be a prime divisor of R associated to I . Then $k(v)$ is generated by the image of a/b over k .

We remark that Theorem 3.2 does not extend, in general, to the case where I is a power of a simple ideal.

Example 3.4. Let $R = k[x, y]_{(x, y)}$ and $m = (x, y)R$ with an algebraically closed field k . Let $I = m^2$. Then $T(I) = \{o\}$, where o is the order valuation, i.e., the Rees valuation of I is the m -adic prime divisor of R , $R[x/y]_{mR[x/y]}$. (x^2, y^2) is a minimal reduction of I . Since $m = (x, y)$ and $T(m) = \{o\}$, by Theorem 3.2, $k(o) = k(\theta)$, where θ is the image of x/y in $R[x/y]/mR[x/y]$. Since $o(x^2) = o(y^2) = 2$, we have that x^2/y^2 is unit in $R[x/y]_{mR[x/y]}$, and hence the image of x^2/y^2 in $R[x/y]/mR[x/y]$ is θ^2 . But $k(\theta^2) \subseteq k(\theta)$.

Lemma 3.5 (Lipman, [2]). Let I be an m -primary integrally closed ideal in (R, m, k) with an algebraically closed field k . Assume that $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$ is the

unique factorization of I as a product simple integrally closed ideals $I_1, \dots, I_l$. Then

$$e(I) = \sum_{i=1}^l \mu_i v_i(I),$$

where v_i is the prime divisor associated to I_i for $i = 1, \dots, l$.

Lemma 3.6 ([4], Theorem 1.1). *Let (A, n) be a local ring with infinite residue field and $J = (x_1, \dots, x_d)A$ an ideal generated by a system of parameters of A . Set*

$$T = A[x_1/x_d, \dots, x_{d-1}/x_d]_{nA[x_1/x_d, \dots, x_{d-1}/x_d]}.$$

Then $e(J) = e(JT)$.

Theorem 3.7. *Let I be an m -primary integrally closed ideal in (R, m, k) with an algebraically closed field k . Assume that $T(I) = \{v_1, \dots, v_l\}$. Then the following conditions are equivalent.*

- (1) $I = I_1 \cdots I_l$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in R .
- (2) For any reduction (a, b) of I , $k(v_i) = k((a/b)_i^*)$, where $(a/b)_i^*$ is the image of a/b in $k(v_i)$ for $i = 1, \dots, l$.
- (3) For some reduction (a, b) of I , $k(v_i) = k((a/b)_i^*)$, where $(a/b)_i^*$ is the image of a/b in $k(v_i)$ for $i = 1, \dots, l$.

Proof. (1) $\Rightarrow$ (2) : This follows immediately from Theorem 3.2.

(2) $\Rightarrow$ (3) : It is clear.

(3) $\Rightarrow$ (1) : Assume that (3) is true. Let $J = (a, b)$. By Zariski's Unique Factorization and One-to-One Correspondence Theorems, $I = I_1^{\mu_1} \cdots I_l^{\mu_l}$, where $I_1, \dots, I_l$ are distinct simple m -primary integrally closed ideals in R and $\mu_i \geq 1$ for $i = 1, \dots, l$, and $\text{Min}(mR[I/b]) = \{q_1, \dots, q_l\}$ and, upon reordering, v_i is the discrete valuation of $R[I/b]_{q_i}$ for $i = 1, \dots, l$. $mR[a/b]$ is a prime ideal in $R[a/b]$ of $ht(mR[a/b]) = 1$ since a, b are a regular sequence on R . Set $T = R[a/b]_{mR[a/b]}$. Then we have

$$\begin{aligned} e(I) &= \sum_{i=1}^l \mu_i v_i(I) && \text{by Lemma 3.5} \\ &= e(J) && (J \text{ is a reduction of } I) \\ &= e(JT) && \text{by Lemma 3.6.} \end{aligned}$$

Let T' be the integral closure of T . Then $T' = R[I/b]_{R[a/b] - mR[a/b]}$ is an one-dimensional Noetherian normal domain since $R[a/b] \subseteq R[I/b]$ is integral and $R[I/b]$ is normal. By Corollary of the Krull-Akizuki Theorem ([6], Corollary of Theorem 11.7), there are just a finite number of maximal ideals of T' lying over mT .

Claim. $mR[a/b] = q_i \cap R[a/b]$ for $i = 1, \dots, l$.

Let $Q_i = q_i \cap R[a/b]$ for $i = 1, \dots, l$. Then $Q_i \supseteq mR[a/b]$. It is enough to show that $ht(Q_i) = 1$ for $i = 1, \dots, l$. Suppose that $ht(Q_i) = 2$. Then $R[a/b]/Q_i \cong k[X]/Q'_i$, where Q'_i is the image of Q_i in $k[X]$, since $Q_i \supseteq mR[a/b]$. Hence the 0-dimensional domain $R[a/b]/Q_i$ is a finitely generated k -algebra. By the Nullstellensatz ([1], Corollary 5.24), $R[a/b]/Q_i$ is a finite algebraic extension of k . Since k is an algebraically closed field, $R[a/b]/Q_i = k$. But $(a/b)^* \in R[a/b]/Q_i$, where $(a/b)^*$ is the image of a/b in $R[a/b]/Q_i$, which is a contradiction since $(a/b)^*$ is transcendental over k . The proof of the claim is complete.

By the above claim, $q_1T', \dots, q_lT'$ are all the maximal ideals of T' lying over mT and

$$T' = \bigcap_{i=1}^l (T')_{q_iT'} = \bigcap_{i=1}^l R[I/b]_{q_i}.$$

By the projection formula ([8], Corollary 1, p. 299),

$$e(JT) = \sum_{i=1}^l [T'/q_iT' : T/mT] e(J(T')_{q_iT'}).$$

Since $R[a/b]/mR[a/b] \subseteq R[I/b]/q_i$ have the same quotient fields by the assumption, $T/mT = T'/q_iT'$ for all i . $e(J(T')_{q_iT'}) = v_i(J)$ since $(T')_{q_iT'} = R[I/b]_{q_i}$ is a Rees valuation ring of I with associated valuation v_i for $i = 1, \dots, l$. $v_i(J) = v_i(I)$ for all i since $J \subseteq I$ is a reduction of I . We have

$$e(JT) = \sum_{i=1}^l v_i(I).$$

Hence we have

$$\sum_{i=1}^l \mu_i v_i(I) = \sum_{i=1}^l v_i(I).$$

That is,

$$\sum_{i=1}^l (\mu_i - 1) v_i(I) = 0.$$

Since $v_i(I)$ is positive value and $\mu_i \geq 1$ for all i , $\mu_i = 1$ for all i . The proof is complete. $\square$

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