SUBLATTICES OF THE BANACH ENVELOPE OF WEAK L^1

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ABSTRACT. We prove that every separable Banach lattice is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .

For a finite nonatomic separable measure space (X, Σ, μ) we study lattice isometric and order isometric embeddings into the Banach envelope $\mathbb{W}$ of the Lorentz space $L(1, \infty)$. It was shown in [4] that L^1 and ℓ^∞ are lattice isometric to closed sublattices of $\mathbb{W}$, and it was shown in [9] that a separable Banach lattice with order continuous norm is lattice isometric to a closed sublattice of $\mathbb{W}$. Using a different approach than in [4] and [9], we show in this paper that *every* separable Banach lattice is lattice isometric to a closed sublattice of $\mathbb{W}$ and that L^∞ as well as Weak L^p , $1 < p < \infty$, of a separable measure space is lattice isometric to a closed sublattice of $\mathbb{W}$. See also [7], in which the first author has given a characterization of the dual space of $L(1, \infty)$.

Let (X, Σ, μ) be a measure space. The Lorentz space $L(1, \infty)$ consists of all (equivalence classes of) μ -measurable functions f on X for which

$$\|f\| = \sup_{s>0} s\mu\{|f| > s\} < \infty.$$

This space with the topology defined by the quasinorm $\|\cdot\|$ is called Weak L^1 . Let U be the convex hull of $V = \{f : \|f\| \leq 1\}$ and let $\|\cdot\|$ be the Minkowski functional of U . Since V is solid, i.e., $f \in V, |g| \leq |f|$ implies $g \in V$, U is solid. Hence $\|\cdot\|$ is a lattice seminorm on $L(1, \infty)$ and $\|\cdot\| \leq \|\cdot\|$. Let I be the ideal $\{f : \|f\| = 0\}$. Then $L(1, \infty)/I$ with the quotient order and the norm $\|f + I\| = \|f\|$ is a normed vector lattice. Its completion $\mathbb{W}$ is a Banach lattice, which is called the Banach envelope of Weak L^1 . By identifying f with $f + I$, we can identify Weak L^1 with a dense sublattice of $\mathbb{W}$.

It has been shown in [1] and [2] that in the nonatomic case

$$\|f\| = \limsup_{q/p \rightarrow \infty} \int_{p < |f| \leq q} |f| d\mu / \ln \frac{q}{p}.$$

We will only consider Weak L^1 and its Banach envelope in the case where (X, Σ, μ) is a finite nonatomic separable measure space. Thus, we can restrict ourselves to the case $X = (0, 1]$ and $\mu = \lambda$, where λ denotes Lebesgue measure on

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$(0, 1]$. Throughout this paper, λ denotes Lebesgue measure on $(0, 1]$ and L^1 denotes the space $L^1((0, 1], \lambda)$.

Let E and F be Banach lattices. A linear operator T from E into F is called a *lattice homomorphism* if $|Tx| = |x|$ for every $x \in E$. If, in addition, $\|Tx\| = \|x\|$ for all $x \in E$, then T is called a *lattice isometry*.

1

We start with embeddings of L^1 and $L^\infty(0, \infty)$ into Weak L^1 and its envelope $\mathbb{W}$. We use the unit square $X = (0, 1] \times (0, 1]$ with Lebesgue measure μ rather than the unit interval (cf. g) of the proof of Theorem 1 below).

Proposition 1. *For $f \in L^1$ define Tf by*

$$(Tf)(s, t) = f(t)/s, \quad (s, t) \in X.$$

Then T is a lattice isometry from L^1 into the Banach envelope of Weak L^1 on X and a lattice “isometry” into the quasinormed space Weak L^1 .

Proof. An easy calculation shows that

$$\|f\| = \sup_{p>0} p \int_0^1 \int_0^{\frac{|f(t)|}{p} \wedge 1} ds dt = \|f\|$$

and that

$$\|Tf\| = \limsup_{\substack{q/p \rightarrow \infty \\ 0 < p < q}} \int_0^1 \int_{\frac{|f(t)|}{q} \wedge 1}^{\frac{|f(t)|}{p} \wedge 1} \frac{|f(t)|}{s} ds dt \Bigg/ \ln \frac{q}{p} = \|f\|.$$

□

Proposition 2. *There exists a lattice homomorphism T from the Banach lattice $L^\infty(0, \infty)$ into Weak L^1 with*

$$\|Tf\| = \|Tf\| = \|f\|$$

for all $f \in L^\infty$.

Proof. Let $\{(u_n, v_n] : u_n < v_n\}$ be an enumeration of the subintervals of $(0, \infty)$ with rational endpoints. Define $\psi_n : (0, 1] \rightarrow (u_n, v_n]$ by $\psi_n(t) = u_n + (v_n - u_n)t$. For $f \in L^\infty$ let $f_n = f \circ \psi_n$. Then

$$\|f\| = \limsup \int |f \circ \psi_n| d\lambda.$$

Choose $0 < a_n < b_n \leq 1$ with $b_{n+1} < a_n$ and $\lim \frac{b_n}{a_n} = \infty$. For $f \in L^\infty$ and $0 < s, t \leq 1$ define

$$(Tf)(s, t) = \begin{cases} \frac{f_n(t)}{s} & \text{for } a_n \leq s \leq b_n, \\ 0 & \text{for } s \notin \cup [a_n, b_n]. \end{cases}$$

Suppose that $f \in L^\infty$ and that $\|f\| = 1$. Then $|(Tf)(s, t)| \leq 1/s$ and hence $\|Tf\| \leq 1$. Fix $0 < \delta < 1$ and let $p_n = \delta/b_n$ and $q_n = 1/a_n$. If $a_n \leq s \leq b_n$ and

$\delta < |f(t)|$, then $p_n < |(Tf)(s, t)| \leq q_n$. Hence

$$\begin{aligned} \int_{p_n < |Tf| \leq q_n} |Tf| d\mu &\geq \int_{|f_n| > \delta} \int_{a_n}^{b_n} \frac{|f_n(t)|}{s} ds d\lambda(t) \\ &= \int_{|f_n| > \delta} |f_n| d\lambda \left(\ln \frac{q_n}{p_n} + \ln \delta \right). \end{aligned}$$

Therefore, $\|Tf\| \geq \limsup \int_{|f_n| > \delta} |f_n| d\lambda$. Since $\limsup \int |f_n| d\lambda = 1$ and $|f_n| \leq 1$, there exists a subsequence (f_{n_i}) which converges λ -uniformly to the function 1. Hence $\limsup \int_{|f_n| > \delta} |f_n| d\lambda = 1$. Consequently, $1 \leq \|Tf\|$, and thus $\|Tf\| = \|Tf\| = 1$. $\square$

In order to embed other Banach lattices into $\mathbb{W}$ a somewhat more elaborate construction is needed. We will construct lattice homomorphisms from nonclosed sublattices of $(\sum \oplus L^1)_{\ell^\infty}$ into Weak L^1 which are isometries for $\|\cdot\|$ but which are not continuous for $\|\cdot\|$ (see Remark 1 below), in contrast to the situation of Propositions 1 and 2.

At first we need a lemma which actually characterizes Weak L^1 functions on any measure space.

Lemma 1. *Let (X, Σ, μ) be a measure space and let f be measurable, f finite a.e. If there exist constants γ and $K, K > 1$, such that $\int_{K^i < |f| \leq K^{i+1}} |f| d\mu \leq \gamma$ for all $i \in \mathbb{Z}$, then $f \in L(1, \infty)$.*

Proof. Let $A_i = \{t : K^i < |f(t)| \leq K^{i+1}\}$. The condition implies that $K^i \mu(A_i) \leq \gamma$. Let $s > 0$ be given. Then for some $j \in \mathbb{Z}, K^j \leq s < K^{j+1}$. Therefore, $s\mu\{|f| > s\} \leq K^{j+1}\mu\{|f| > K^j\} \leq K^{j+1} \sum_{i=j}^{\infty} \mu(A_i) \leq \gamma K^2 / (K - 1)$. Hence, $f \in L(1, \infty)$. $\square$

Theorem 1. *Let $E = (\sum \oplus L^1)_{\ell^\infty}$ and let $M_i \geq 1$ for $i \in \mathbb{N}$. Then the closure of the sublattice $G = \{(h_i) \in E : \sup M_i^{-1} \|h_i\|_\infty < \infty\}$ is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. a) Choose $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $i \in \mathbb{N}$ the set $\varphi^{-1}(\{i\})$ is infinite. Let $d_n = \max_{1 \leq j \leq n} M_{\varphi(j)}$. Then the operator $S : (h_i) \mapsto (f_n)$, where $f_n = h_{\varphi(n)}$, is a lattice isometry from E into itself which maps G into

$$F = \{(f_n) : \sup d_n^{-1} \|f_n\|_\infty < \infty\}.$$

Let $X = (0, 1] \times (0, 1]$ with Lebesgue measure μ . We will construct a lattice homomorphism T from F into the Banach envelope of Weak L^1 on X with $\|T\| \leq 1$ and $\|Tf\| \geq \|f\|$ whenever $f \in SG$.

b) Choose $a_n > 0$ and $b_n > 0$ such that

- i) $b_1 = 1$,
- ii) $a_{n+1} < b_{n+1} < a_n < b_n$,
- iii) $\lim \frac{b_n}{a_n} = \infty$,
- iv) $\ln \frac{a_l}{b_k d_k} \geq \sum_{i=l}^k \ln \frac{b_i}{a_i}$ for $l < k$.

We denote the set $\{(s, t) : a_n \leq s \leq b_n, 0 < t \leq 1\}$ by A_n .

c) For $f = (f_n) \in F$ and $0 < s, t \leq 1$ define

$$(Tf)(s, t) = \begin{cases} \frac{f_n(t)}{s} & \text{for } (s, t) \in A_n, \\ 0 & \text{for } (s, t) \notin \cup A_n. \end{cases}$$

Obviously, T defines a lattice homomorphism from F into the space of measurable functions on the square X . We note that $\int_{A_n} |Tf| d\mu = \|f_n\| \ln \frac{b_n}{a_n}$.

d) Let $f = (f_n) \in F$ be given and let $\gamma = \sup d_n^{-1} \|f_n\|_\infty$. Suppose that there exists $\delta > 0$ such that $f_n(t) \geq \delta$ for all n and all t . Fix p and q , $0 < p < q$, and let $A = \{(s, t) : p < Tf(s, t) \leq q\}$. We claim that

$$(\dagger) \quad \int_A Tf d\mu \leq \|f\| \max \left\{ \ln \frac{q}{p}, \ln \frac{q\gamma}{p\delta} \right\}.$$

We may assume that A is not empty. Let $B_n = A \cap A_n$. Then $A = \cup B_n$. If $(s, t) \in B_n$, then $\delta/q \leq f_n(t)/q \leq s < f_n(t)/p \leq \gamma d_n/p$. Let l and k be the minimum and maximum respectively of $\{n : B_n \neq \emptyset\}$. We consider two cases:

1) If $l = k$, let $a(t) = \min\{\max\{a_k, \frac{f_k(t)}{q}\}, b\}$, $b(t) = \max\{\min\{b_k, \frac{f_k(t)}{p}\}, a\}$; then

$$\begin{aligned} \int_A Tf d\mu &= \int_0^1 \int_{a(t)}^{b(t)} \frac{f_k(t)}{s} ds dt \\ &\leq \int_0^1 \int_{\frac{f_k(t)}{q}}^{\frac{f_k(t)}{p}} \frac{f_k(t)}{s} ds dt \\ &= \|f_k\| \ln \frac{q}{p}. \end{aligned}$$

2) If $l < k$, then $\delta/q \leq b_k \leq a_l \leq \gamma d_l/p \leq \gamma d_k/p$. Hence $a_l/b_k d_k \leq q\gamma/p\delta$. Therefore,

$$\begin{aligned} \int_A Tf d\mu &\leq \sum_{n=l}^k \int_{B_n} Tf d\mu \leq \sum_{n=l}^k \|f_n\| \ln \frac{b_n}{a_n} \\ &\leq \|f\| \sum_{n=l}^k \ln \frac{b_n}{a_n} \leq \|f\| \ln \frac{a_l}{b_k d_k} \leq \|f\| \ln \frac{q\gamma}{p\delta}. \end{aligned}$$

This proves $(\dagger)$. It follows easily from Lemma 1 that $Tf \in L(1, \infty)$. Moreover,

$$\begin{aligned} \|Tf\| &= \limsup_{q/p \rightarrow \infty} \int_A |Tf| d\mu / \ln \frac{q}{p} \\ &= \limsup_{q/p \rightarrow \infty} \int_A Tf d\mu / \ln \frac{q}{p} \\ &\leq \|f\|. \end{aligned}$$

e) Let $g = (g_n) \in F$ be arbitrary. Choose $\eta > 0$ and let $f_n = (|g_n| + \eta)$. Then $f = (f_n) \in F$ and satisfies the assumptions of d). Hence $Tf \in L(1, \infty)$ and $\|Tf\| \leq \|f\| = \|g\| + \eta$. Since $|Tg| \leq Tf$, $Tg \in L(1, \infty)$ and $\|Tg\| \leq \|g\| + \eta$. Since $\eta > 0$ is arbitrary, $\|Tg\| \leq \|g\|$. This shows that T maps F into Weak L^1 and that $\|Tf\| \leq \|f\|$ for every $f \in F$.

f) Let $f = (f_n) \in F$ be given. Fix α and β , $0 < \alpha < \beta$, and let $p_n = \alpha/b_n$ and $q_n = \beta/a_n$. Then

$$\begin{aligned} \int_{p_n < |Tf| \leq q_n} |Tf| d\mu &\geq \int_{\alpha < |f_n(t)| \leq \beta} \int_{a_n}^{b_n} \frac{|f_n(t)|}{s} ds dt \\ &= \left(\int_{\alpha < |f_n(t)| \leq \beta} |f_n(t)| dt \right) \ln \frac{b_n}{a_n}. \end{aligned}$$

Since q_n/p_n tends to ∞ ,

$$\|Tf\| \geq \limsup \int_{\alpha < |f_n(t)| \leq \beta} |f_n(t)| dt .$$

If $f = Sh$, where $h = (h_i) \in G$, then for every i there exist infinitely many n 's with $f_{\varphi(n)} = h_i$. Therefore,

$$\|TSh\| \geq \int_{\alpha < |h_i(t)| \leq \beta} |h_i(t)| dt .$$

Consequently, $\|TSh\| \geq \sup \|h_i\| = \|h\|$.

g) Finally, let $\psi : (0, 1] \rightarrow X$ be a measure-preserving transformation. Then $U : f \mapsto (TSF) \circ \psi$ is a lattice isometry from G into the Banach envelope of Weak L^1 on $(0, 1]$ which can be extended to a lattice isometry from the closure of G into $\mathbb{W}$. $\square$

Corollary 1. *The sublattice $H = \{(f_n) : \{f_n\} \text{ is relatively weakly compact in } L^1\}$ of $(\sum \oplus L^1)_{\ell^\infty}$ is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. It is easily checked that H is the closure of the sublattice $F = \{(f_i) : \sup \|f_i\|_\infty < \infty\}$ of $(\sum \oplus L^1)_\ell^\infty$. Now the Corollary follows from Theorem 1 with $M_i = 1$. $\square$

Corollary 2. *For every sequence (k_i) of natural numbers, the Banach lattice $(\sum \oplus \ell^1(k_i))_{\ell^\infty}$ is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. Clearly, for every i there exists a lattice isometry T_i from $\ell^1(k_i)$ into L^1 with $\|T_i x\|_\infty \leq k_i \|x\|$ for all $x \in \ell^1(k_i)$. Hence, $T : (x_i) \mapsto (T_i x_i)$ is a lattice isometry from $(\sum \oplus L^1)_{\ell^\infty}$ into the sublattice $\{(f_i) : \sup k_i^{-1} \|f_i\|_\infty < \infty\}$ of $(\sum \oplus L^1)_{\ell^\infty}$. $\square$

Remark 1. The map T from F into Weak L^1 in the proof of Theorem 2 is not $\|\cdot\|$ -continuous (this follows from Theorem 5 below). However, the restriction to any sublattice F_1 of F will be continuous if F_1 is closed in $(\sum \oplus L^1)_{\ell^\infty}$ (indeed, every positive linear map from a Banach lattice into $(\text{Weak } L^1, \|\cdot\|)$ is continuous). Consequently, if G_1 is a sublattice of the space G in Theorem 2 and G_1 is closed in $(\sum \oplus L^1)_{\ell^\infty}$, then the restriction of TS is a topological isomorphism from G_1 into $(\text{Weak } L^1, \|\cdot\|)$. In particular, the space $(\sum \oplus \ell^1(k_i)_{\ell^\infty})$ is lattice isomorphic to a closed sublattice of $(\text{Weak } L^1, \|\cdot\|)$.

2

A positive element u of a Banach lattice E is called a *quasi-interior point* if the interval $[0, u] = \{x : 0 \leq x \leq u\}$ is total in E . Every separable Banach lattice has a quasi-interior point.

Every Banach lattice is lattice isometric to a closed sublattice of an ℓ^∞ -product of AL-spaces ([5], Lemma 3.4). We will show that if E is a separable Banach lattice or if E is the dual of a separable Banach lattice and has a quasi-interior point, then E is lattice isometric to a closed sublattice of the Banach lattice H of Corollary 1.

Lemma 2. *Let E be a separable Banach lattice, let $0 \leq u \in E$ be a quasi-interior point, and let $0 < x' \in E'$. Then there exists a lattice homomorphism T from E into L^1 such that $\|Tx\| = \langle x, x' \rangle$ for every $x \in E$ and $Tu = \langle u, x' \rangle 1$.*

Proof. Let I be the closed ideal $\{x : \langle x, x' \rangle = 0\}$ and let F be the completion of E/I with the norm $\|x+I\| = \langle x, x' \rangle$. Then F is an AL-space and the canonical map φ from E into F is a lattice homomorphism with $\|\varphi(x)\| = \langle x, x' \rangle$ for every $x \in E$. Thus, φ is continuous and therefore F is separable. Moreover, $v = \langle u, x' \rangle^{-1} \varphi(u)$ is a weak order unit of F with $\|v\| = 1$. Hence by Kakutani's theorem there is a measure space (S, Σ, ν) , $\nu(s) = 1$, and a lattice isometry T_1 from F onto $G = L^1(S, \Sigma, \nu)$ which maps v onto the function 1. Since G is separable, there is a lattice isometry T_2 from G into L^1 with $T_2 1 = 1$. Thus, $T = T_2 \circ T_1 \circ \varphi$ is the desired lattice homomorphism from E into L^1 . $\square$

Theorem 2. *Every separable Banach lattice E is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. We may assume that $E \neq \{0\}$. Since E is separable, E has a quasi-interior point u . Also, there is a normalized sequence $(x'_i) \subset E'$, $x'_i \geq 0$, such that $\|x\| = \sup \langle x, x'_i \rangle$ for every $x \in E$. By Lemma 2, there exist lattice homomorphisms $T_i : E \rightarrow L^1$ such that $\|T_i x\| = \langle x, x'_i \rangle$ for every $x \in E$ and $T_i u = \langle u, x'_i \rangle 1$. Obviously, $x \mapsto (T_i x)$ is then a lattice isometry from E into $(\sum \oplus L^1)_{\ell^\infty}$. If $0 \leq x \leq u$, then $0 \leq T_i x \leq \langle u, x'_i \rangle 1 \leq \|u\| 1$, since $\sup \|x'_i\| \leq 1$, and hence $\{T_i x\}$ is relatively weakly compact. Since the interval $[0, u]$ is total in E , T maps E into the sublattice H of $(\sum \oplus L^1)_{\ell^\infty}$ of all relatively weakly compact sequences. Now it follows from Corollary 1 that E is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 . $\square$

Let E be a Banach lattice and let $0 \leq x \in E$. The linear hull of the order interval $[0, x]$ is called the *principal ideal generated by x* and denoted by E_x .

Lemma 3. *Let E be a separable Banach lattice and let $0 < x \in E$ and $0 \leq v' \in E'$. Then there exists a lattice homomorphism T from the closure G of the principal ideal E'_x into L^1 with $Tv' = \langle x, v' \rangle 1$ and $\|Ty'\| = \langle x, |y'| \rangle$ for every $y' \in G$.*

Proof. By Kakutani's representation theorem for AM-spaces with unit, there exist a compact space K and a lattice isomorphism S from $C(K)$ onto the principal ideal E_x with $S1 = x$. Since S is a lattice homomorphism and preserves order intervals, S' is a lattice homomorphism from E' into the AL-space $C(K)'$ with $S'[0, v'] = [0, \nu]$ where $\nu = S'v'$ ([5], Proposition 1.2). Clearly, $\|S'x'\| = \langle x, |x'| \rangle$ for every $x' \in E'$. Also, S' maps the ideal G into the band $L^1(K, \nu)$. Since E is separable, $[0, v']$ is separable for the weak* topology. Hence $[0, \nu]$ is also weak*-separable. Now, $[0, \nu]$ is weakly compact, and so weakly separable. Therefore, $L^1(K, \nu)$ is separable. Thus there exists a lattice isometry T_2 from $L^1(K, \nu)$ into L^1 with $T_2 \nu = \langle x, \nu \rangle 1$. Let T_1 be the restriction of S' to G . Then $T = T_2 \circ T_1$ is the desired lattice homomorphism. $\square$

Theorem 3. *If E is a separable Banach lattice, then the closure of any principal ideal in E' is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. Let $0 \leq v' \in E'$ be given and let G be the closure of the principal ideal $E'_{v'}$. Choose $0 \leq x_i \in E$ such that $\|x'\| = \sup \langle x_i, |x'| \rangle$ for every $x' \in E'$. By Lemma

3, there exist lattice homomorphisms T_i from G into L^1 with $T_i v' = \langle x_i, v' \rangle 1$ and $\|T_i x'\| = \langle x_i, |x'| \rangle$ for every $x' \in G$. As in the proof of Theorem 2, it follows easily that $T : x' \mapsto (T_i x')$ is a lattice isometry from G into the sublattice W of $(\sum \oplus L^1)_{\ell^\infty}$ of all relatively weakly compact sequences. Now, Corollary 1 implies that G is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 . $\square$

Corollary 3. *Let E be a separable Banach lattice. If the dual E' has a quasi-interior point, then E' is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Corollary 4. *Let (S, Σ, ν) be a separable measure space. Then the Banach lattice $L^\infty(S, \Sigma, \nu)$ is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Of course, Corollary 4 also follows from Proposition 2.

Let (S, Σ, ν) be a measure space. The space Weak L^p consists of all (equivalence classes of) measurable functions f on S for which

$$\sup_{s>0} s(\nu\{|f| > s\})^{1/p} < \infty.$$

If $1 < p < \infty$, then Weak L^p with the norm

$$\|f\| = \sup_{0 < \nu(B) < \infty} \nu(B)^{1/p-1} \int_B |f| d\nu$$

is a dual Banach lattice, which we denote by $L(p, \infty, \nu)$. Except in the finite dimensional case, this space does not have a quasi-interior point.

Theorem 4. *Let (S, Σ, ν) be a separable measure space and let $1 < p < \infty$. Then Weak L^p is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 .*

Proof. At first we consider the real interval $(0, \alpha)$, $0 < \alpha \leq \infty$, with Lebesgue measure μ . Since the predual $L(p/(p-1), 1)$ is separable, there is a sequence (B_n) of measurable subsets of $(0, \alpha)$ such that

$$\|f\| = \sup_n \mu(B_n)^{1/p-1} \int_{B_n} |f| d\mu, \quad f \in L(p, \infty, \mu).$$

For each n there exists a bijection ψ_n from $(0, 1]$ onto B_n such that $A \subset (0, 1]$ is measurable if and only if $\psi_n(A)$ is measurable and such that $\mu(\psi_n(A)) = \lambda(A)\mu(B_n)$. An easy calculation shows that for every measurable subset $A \subset (0, 1]$,

$$\begin{aligned} \mu(B_n)^{1/p} \int_A |f \circ \psi_n| d\lambda &= \mu(B_n)^{1/p-1} \int_{\psi_n(A)} |f| d\mu \\ &\leq \mu(B_n)^{1/p-1} \mu(\psi_n(A))^{1-1/p} \|f\| \\ &\leq \lambda(A)^{1-1/p} \|f\|. \end{aligned}$$

This shows that $\{\mu(B_n)^{1/p} f \circ \psi_n\}$ is relatively weakly compact in L^1 . Also, the first equation with $A = (0, 1]$ shows that $f \mapsto (\mu(B_n)^{1/p} f \circ \psi_n)$ is a lattice isometry from Weak L^p into $(\sum \oplus L^1)_{\ell^\infty}$. It follows from Corollary 1 that Weak L^p is lattice isometric to a closed sublattice of the Banach envelope of Weak L^1 . Now let (S, Σ, ν) be separable and σ -finite and let $\alpha = \nu(S)$. Then (S, Σ, ν) is isomorphic to a σ -subalgebra Σ_1 of the real interval $(0, \alpha)$ with Lebesgue measure μ . It is

clear that $L(p, \infty, \nu)$ is lattice isomorphic to the sublattice of all Σ_1 -measurable functions of $L(p, \infty, \mu)$. It follows from Lemma 8 of [6] that this isomorphism is an isometry. $\square$

Remark 2. Let $1 < p < \infty$ and let $1/p + 1/q = 1$. The inclusion map from $L^p(0, 1]$ into Weak L^p on $(0, 1]$ is a lattice isomorphism but not an isometry. For $f \in L^p$, define Tf by

$$(Tf)(s, t) = qf(t)/s^{1/p}, \quad (s, t) \in X = (0, 1] \times (0, 1].$$

Let $\varphi : (0, 1] \rightarrow X$ be a measure-preserving transformation. Then $f \mapsto (Tf) \circ \varphi$ is a lattice isometry from L^p into the Lorentz space $L(p, \infty, \lambda)$.

3

It follows from Theorem 2 that there is a lattice isometry from $(\sum \oplus L^1)_{c_0}$ into the Banach envelope of Weak L^1 . However, no such embedding factors through Weak L^1 as we will see. This shows that the operator T in the proof of Theorem 2 is not a continuous operator into the quasinormed space Weak L^1 .

Lemma 4. *Let $0 < f \in \text{Weak } L^1$ with $\|f\| > 1$. Then for every $\alpha > 1$ there exists $p > 0$ such that*

$$\int_{p < f \leq \alpha p} f d\lambda \geq \ln \alpha.$$

Proof. Choose $k \in \mathbb{N}$ such that $\|f\| > (k+1)/k$. Then there exist $0 < r < q$ such that $\alpha^k \leq q/r$ and $(k+1)/k \ln(q/r) \leq \int_{r < f \leq q} f d\lambda$. Let $m = \lceil \ln(q/r)/\ln \alpha \rceil$. Then $m \geq k$ and

$$\begin{aligned} (m+1) \ln \alpha &\leq \left(1 + \frac{1}{k}\right) m \ln \alpha \leq \left(1 + \frac{1}{k}\right) \ln \frac{q}{r} \\ &\leq \int_{r < f \leq q} f d\lambda \leq \int_{r < f \leq \alpha^{m+1} r} f d\lambda = \sum_{i=0}^m \int_{\alpha^i r < f \leq \alpha^{i+1} r} f d\lambda \\ &\leq (m+1) \max \left\{ \int_{\alpha^i r < f \leq \alpha^{i+1} r} f d\lambda \right\}. \end{aligned}$$

Hence, for some j , $\ln \alpha \leq \int_{\alpha^j r < f \leq \alpha^{j+1} r} f d\lambda$. Now let $p = \alpha^j r$. $\square$

Lemma 5. *Let T be a lattice homomorphism from L^1 into Weak L^1 and let $0 \leq f \in L^1$ and $p < q$ be given. Suppose that $\int_{p < Tf \leq q} Tf d\lambda > 0$. Then for every $\gamma \geq 1$ there exists $0 \leq g \in L^1$ with $\|g\| \leq \|f\|$ and*

$$\int_{\gamma p < Tg \leq \gamma q} Tg d\lambda = \int_{p < Tf \leq q} Tf d\lambda.$$

Proof. The assertion is trivial if $\gamma = 1$. So we assume that $\gamma > 1$. Let $D = \{p < Tf \leq q\}$. For every λ -measurable set A let $\nu(A) = \int_D T(f\chi_A) d\lambda$. Since T is a continuous map from L^1 into Weak L^1 (indeed, $\|f_i\| \leq 2^{-2i}$ implies $|Tf_n| \leq 2^{-n}T(\sum 2^i |f_i|)$), $T(f\chi_A)$ tends to zero in measure as $\lambda(A)$ tends to zero. By the dominated convergence theorem $\int_D T(f_A) d\lambda$ tends to zero as $\lambda(A)$ tends to zero. Hence ν is a measure which is absolutely continuous with respect to λ .

Therefore, there exists $0 \leq h_1 \in L^1$ with $\nu(A) = \int_A h_1 d\lambda$. It is clear that h_1 vanishes λ -almost everywhere on the set $\{f = 0\}$. Let $h(t) = h_1(t)/f(t)$ if $f(t) > 0$

and $h(t) = 0$ elsewhere. Then $\nu(A) = \int_A hf d\lambda$ for every measurable set A . Let $\beta = \inf\{\alpha : \int_{h>\alpha} hf d\lambda \leq \gamma^{-1}||\nu||\}$. Then $\int_{h>\beta} hf d\lambda \leq \gamma^{-1}||\nu|| \leq \int_{h\geq\beta} hf d\lambda$. There exists a measurable set B with $\{h > \beta\} \subset B \subset \{h \geq \beta\}$ and $\int_B hf d\lambda = \gamma^{-1}||\nu||$. Then $\beta \int_B f d\lambda \leq \gamma^{-1}||\nu||$ and $(1 - \gamma^{-1})||\nu|| \leq \beta(||f|| - \int_B f d\lambda)$, and so $\int_B f d\lambda / (||f|| - \int_B f d\lambda) \leq \gamma^{-1} / (1 - \gamma^{-1})$. This easily implies that $\int_B f d\lambda \leq \gamma^{-1}||f||$. Let $g = \gamma f \chi_B$. Then $||g|| \leq ||f||$ and

$$\int_{p < Tg \leq q} Tg d\lambda = \gamma \nu(B) = \gamma \int_B hf d\lambda = ||\nu|| = \int_{p < Tf \leq q} Tf d\lambda.$$

Since T is a lattice homomorphism, $T(f\chi_B)$ and $T(f\chi_{B^c})$ are disjoint. Hence

$$\int_{p < T(f\chi_B) \leq q} T(f\chi_B) d\lambda = \int_{p < Tf \leq q} T(f\chi_B) d\lambda.$$

Therefore

$$\int_{\gamma p < Tg \leq \gamma q} Tg d\lambda = \gamma \int_{p < Tf \leq q} T(f\chi_B) d\lambda = \int_{p < Tf \leq q} Tf d\lambda.$$

□

Theorem 5. *Let T be a lattice homomorphism from $(\sum \oplus L^1)_{c_0}$ into Weak L^1 . Then T as a map into the Banach envelope of Weak L^1 factors through L^1 .*

Proof. For $i \in \mathbb{N}$ define $\varphi_i : L^1 \rightarrow (\sum \oplus L^1)_{c_0}$ by $\varphi_i(f) = (f_n)$, where $f_n = f$ if $n = i$ and $f_n = 0$ otherwise, and let $T_i = t \circ \varphi_i$. We may assume that $||T_i|| > 0$ for all i . Suppose that $\sum ||T_i|| = \infty$. Then there exist $0 \leq f_i \in L^1$, $\lim ||f_i|| = 0$, with $\sum ||T_i f_i|| = \infty$. Let $\{A_n\}$ be a partition of $\mathbb{N}$ into finite subsets such that $\sum_{i \in A_n} ||T_i f_i|| > n$ for every n .

By Lemma 4, for every $i \in A_n$ we can choose $p_i > 0$ such that $\int_{p_i < T_i f_i \leq e^n p_i} T_i f_i > n ||T_i f_i|| (1 - n^{-1})$. Let $r_n = \max\{p_i : i \in A_n\}$. By Lemma 5, for $i \in A_n$ there exists $0 \leq g_i \in L^1$, $||g_i|| \leq ||f_i||$, such that $\int_{r_n < T_i g_i \leq e^n r_n} T_i g_i > n ||T_i f_i|| (1 - n^{-1})$. Let $g = (g_i)$. Then $g \in F$, $Tg = \sum T_i g_i$, and the functions $T_i g_i$ are pairwise disjoint. Therefore,

$$\begin{aligned} \int_{r_n < Tg \leq e^n r_n} Tg d\mu &= \sum \int_{r_n < T_i g_i \leq e^n r_n} T_i g_i d\mu \\ &\geq \sum_{i \in A_n} \int_{r_n < T_i g_i \leq e^n r_n} T_i g_i d\mu \geq n(n-1). \end{aligned}$$

Obviously, this contradicts the fact that

$$\limsup_{q/p \rightarrow \infty} \int_{p < Tg \leq q} Tg d\mu / \ln \frac{q}{p} < \infty.$$

Thus, $\sum ||T_i|| < \infty$. Let G be the completion of F for the norm $||(f_i)||_1 = \sum ||T_i|| ||f_i||$. Then G is isometric to L^1 and T as a map into $\mathbb{W}$ factors through G . □

A positive linear operator T from a Banach lattice E into a Banach lattice F is called an *order isometry* if $Tx \geq 0$ implies $x \geq 0$ and if $||Tx|| = ||x||$ for all $x \in E$. It follows from Theorem 5 that no lattice isometry from $(\sum \oplus L^1)_{\ell^\infty}$ into the Banach envelope of Weak L^1 factors through Weak L^1 by a lattice homomorphism.

However, there is an order isometry from $(\sum \oplus L^1)_{\ell^\infty}$ into the Banach envelope of Weak L^1 .

Theorem 6. *The Banach lattice $(\sum \oplus L^1)_{\ell^\infty}$ is order isometric to a closed subspace of the Banach envelope of Weak L^1 .*

Proof. For $f \in L^1$ let $P_n f$ be the conditional expectation of f with respect to the σ -subalgebra generated by the intervals $((j-1)2^{-n}, j2^{-n}]$. We may also consider P_n as a positive operator from L^1 into $\ell^1(2^n)$. Since the P_n 's are positive contractions and $P_n f$ converges to f in L^1 , the map $f \mapsto (P_n f)$ is an order isometry from L^1 into $(\sum \oplus L^1)_{\ell^\infty}$. Then also $T : (f_i) \mapsto (P_n f_i)$ is an order isometry from $E = (\sum \oplus L^1)_{\ell^\infty}$ into $F = (\sum \oplus \ell^1(k_{(i,n)}))_{\ell^\infty}$, where $k_{(i,n)} = 2^n$ for every $(i, n) \in \mathbb{N} \times \mathbb{N}$. By Corollary 2, there is a lattice isometry S from F into the Banach envelope of Weak L^1 . Hence $S \circ T$ is an order isometry from E into the Banach envelope of Weak L^1 . $\square$

Corollary 5. *Let E be a separable Banach lattice and let $0 < x' \in E'$. Then the band generated by x' in E' is lattice isometric to a closed sublattice of $(\sum \oplus L^1)_{\ell^\infty}$.*

Proof. A slight modification of the proofs of Lemma 3 and Theorem 3 shows that the band generated by x' is lattice isometric to a closed sublattice of $(\sum \oplus L^1)_{\ell^\infty}$. $\square$

Corollary 6. *Let E be a separable Banach lattice. If E' has a weak order unit, then E' is order isometric to a closed subspace of the Banach envelope of Weak L^1 .*

Corollary 7. *Let E be a separable Banach lattice. If E does not contain a closed subspace order isometric to $C(\Delta)$, Δ the Cantor set, then the dual E' is order isometric to a closed subspace of the Banach envelope of Weak L^1 .*

Proof. Since E is separable, it follows easily from Theorem 2 of [8] that E' has a weak order unit if and only if E does not contain a closed subspace order isometric to $C(\Delta)$. $\square$

Remark 3. It follows from Remark 1 that the spaces in Theorem 6 and Corollaries 5, 6, and 7 are order isomorphic (and thus topologically isomorphic) to closed subspaces of $(\text{Weak } L^1, \|\cdot\|)$.

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