EIGENVALUES OF SOME DISTAL FUNCTIONS

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Dedicated to Professor Junji Kato on his sixtieth birthday

ABSTRACT. In this paper we construct distal functions of another type discussed by Salehi (1991). Let $a(t)$ be an almost periodic function with the mean value 0, which has unbounded integral, and Φ a continuous periodic function with the prime period 1. If Φ satisfies some additional condition, then $f(t) = \Phi(\int_0^t a(s) ds)$ is a distal function, which is not almost periodic, and the set of eigenvalues of f is the module of a .

1. PRELIMINARIES

Let X be a metric space with the metric d_X . Z, R and C denote the set of integers, the set of real numbers and the set of complex numbers, respectively. A continuous mapping $\pi : X \times R \rightarrow X$ is said to be a *flow on (a phase space) X* if π satisfies the following two conditions:

(1) $\pi(x, 0) = x$ for $x \in X$.

(2) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for $x \in X$ and $t, s \in R$.

For $A \subset X$ and $B \subset R$ we denote the set $\{\pi(x, t); x \in A, t \in B\}$ by $\pi(A, B)$. The closure of $A \subset X$ is denoted by $\overline{A}$. The orbit of π through $x \in X$ is denoted by $O_\pi(x)$, that is, $O_\pi(x) = \pi(x, R)$. $M \subset X$ is called an *invariant set* of π if $O_\pi(x) \subset M$ for each $x \in M$. The restriction of π to an invariant set M of π is denoted by $\pi|M$. A non-empty compact invariant set of π is called a *minimal set* if $\overline{O_\pi(x)} = M$ for each $x \in M$. If X is itself a minimal set of π , we say that π is a *minimal flow on X* . We say that π is *equicontinuous* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(\pi(x, t), \pi(y, t)) < \varepsilon$ for $d_X(x, y) < \delta$ and $t \in R$. π is said to be *distal* if $\inf_{t \in R} \{d_X(\pi(x, t), \pi(y, t))\} > 0$ for $x, y \in X$ ($x \neq y$).

Let π be a minimal flow on a compact metric space X . $\lambda \in R$ is called an *eigenvalue* of π if there exists a continuous mapping χ_λ from X to $K = \{\xi \in C; |\xi| = 1\}$ such that $\chi_\lambda(\pi(x, t)) = \exp(2\pi i \lambda t) \chi_\lambda(x)$ for $(x, t) \in X \times R$. In this case χ_λ is said to be an *eigenfunction belonging to λ* . The set of eigenvalues of π is denoted by $\Lambda(\pi)$. We can easily see that $\Lambda(\pi)$ is a countable subgroup of the additive group R .

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$C(R, C)$ denotes the set of continuous functions from R to C with the compact-open topology. Then $C(R, C)$ is a complete metric space. Define a flow η on $C(R, C)$ by $\eta(f, t) = f_t$ for $(f, t) \in C(R, C) \times R$, where $f_t(s) = f(t + s)$ for $s \in R$. For $f \in C(R, C)$ we denote the hull of f by $H(f)$, that is, $H(f) = \overline{O_\eta(f)} = \overline{\{f_t\}_{t \in R}}$. The restriction of η to $H(f)$ ($f \in C(R, C)$) is denoted by η_f . $L \subset R$ is said to be *relatively dense* if there exists an $l > 0$ such that $[l - t, l + t] \cap L \neq \emptyset$ for each $t \in R$. $f \in C(R, C)$ is said to be an *almost periodic function* if for each $\varepsilon > 0$ there exists a relatively dense set $A_\varepsilon \subset R$ such that $|f(t + \tau) - f(t)| < \varepsilon$ for $t \in R$ and $\tau \in A_\varepsilon$. It is well known that, if f is an almost periodic function, then $H(f)$ is compact and η_f is an equicontinuous minimal flow on $H(f)$. $f \in C(R, C)$ is said to be a *distal function* if $H(f)$ is compact, and if η_f is a distal flow on $H(f)$. It is known ([4], p. 37) that, if f is a distal function, then η_f is a minimal flow on $H(f)$.

Let π and ρ be flows on X and Y , respectively. A continuous mapping h from X to Y is said to be a *homomorphism from π to ρ* if $h(\pi(x, t)) = \rho(h(x), t)$ for $(x, t) \in X \times R$. Furthermore, if h is a homeomorphism from X onto Y , then h is called an *isomorphism from π to ρ* . In this case, we say that π and ρ are isomorphic.

Proposition 1.1. *Let π and ρ be minimal flows on compact metric spaces X and Y , respectively. If π and ρ are isomorphic, then $\Lambda(\pi) = \Lambda(\rho)$.*

Proof. Easy.

2. SOME DISTAL FLOWS

In this section π is an equicontinuous minimal flow on a compact metric space X , ϕ is a continuous mapping from $X \times R$ to R satisfying the condition

$$(*) \quad \phi(x, t + s) = \phi(x, t) + \phi(\pi(x, t), s) \quad \text{for } x \in X \text{ and } t, s \in R,$$

and ρ is a flow on $X \times K$ defined by

$$\rho((x, \xi), t) = (\pi(x, t), \xi \exp(2\pi i \phi(x, t))) \quad \text{for } (x, \xi) \in X \times K \text{ and } t \in R.$$

Proposition 2.1. *If ϕ is not uniformly continuous, then ρ is a distal minimal flow on $X \times K$, which is not equicontinuous, such that $\Lambda(\rho) = \Lambda(\pi)$.*

Proof. See [3], Theorems 4.3 and 4.5.

Remark 2.2. (1) $\lim_{t \rightarrow \infty} \frac{1}{t} \phi(x, t)$ exists for $x \in X$ and this limit is constant.

(2) Assume that $\lim_{t \rightarrow \infty} \frac{1}{t} \phi(x, t) = 0$ for some $x \in X$. Then ϕ is not uniformly continuous if and only if $\phi(x, t)$ is unbounded on R for each $x \in X$.

Proof. (1) See [1], Lemma 2.

(2) We assume that ϕ is not uniformly continuous on $X \times R$. Assume there exists $x_0 \in X$ such that $|\phi(x_0, t)| \leq M$ for $t \in R$. Put $\bar{\phi}(x, t) = \frac{1}{4M} \phi(x, t)$ for $(x, t) \in X \times R$. Then we have $|\bar{\phi}(x_0, t)| \leq \frac{1}{4}$, $\bar{\phi}$ satisfies the condition (*), and it is not uniformly continuous. Define a flow $\bar{\rho}$ on $X \times K$ by $\bar{\rho}((x, \xi), t) = (\pi(x, t), \xi \exp(2\pi i \bar{\phi}(x, t)))$ for $(x, \xi) \in X \times K$ and $t \in R$. By Proposition 2.1 $\bar{\rho}$ must be a minimal flow on $X \times K$. But, since $|\bar{\phi}(x_0, t)| \leq \frac{1}{4}$, $\overline{O_{\bar{\rho}}(x_0, 1)} \neq X \times K$. This is a contradiction. Conversely, if ϕ is uniformly continuous on $X \times R$, there exists a continuous function $\Psi : X \rightarrow R$ such that

$$\phi(x, t) = -\Psi(\pi(x, t)) + \Psi(x) \quad \text{for } (x, t) \in X \times R$$

([1]). Hence $\phi(x, t)$ is bounded on R for each $x \in X$.

Remark 2.3. Let $C(X)$ be a set of real-valued continuous functions on X with the topology of uniform convergence, and let

$$C_0(X) = \{b \in C(X); \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b(\pi(x, s)) ds = 0\}.$$

For $b \in C_0(X)$, put $\phi_b(x, t) = \int_0^t b(\pi(x, s)) ds$ for $(x, t) \in X \times R$. Then ϕ_b satisfies (*) and there exists $b \in C_0(X)$ such that ϕ_b is not bounded on R ([5], p. 362).

Let Φ be a periodic continuous function on R with the prime period 1. For $\tau \in R$, put $\Phi_\tau(t) = \Phi(\tau + t)$ for $t \in R$, and $S = \{\Phi_\tau; \tau \in R\}$ with the topology of uniform convergence. Then we can easily see that S is homeomorphic to K . Define a flow $\tilde{\rho}$ on $X \times S$ by $\tilde{\rho}((x, \Phi_\tau), t) = (\pi(x, t), \Phi_{\tau + \phi(x, t)})$ for $(x, \Phi_\tau) \in X \times S$ and $t \in R$, where ϕ is as in Proposition 2.1.

Proposition 2.4. *If ϕ is not uniformly continuous on $X \times R$, then $\tilde{\rho}$ is a distal minimal flow on $X \times S$ which is not equicontinuous, and $\Lambda(\tilde{\rho}) = \Lambda(\pi)$.*

Proof. Let h be the mapping from $X \times S$ to $X \times K$ defined by

$$h(x, \Phi_\tau) = (x, \exp(2\pi i \tau)).$$

Then h is a homeomorphism and we have

$$\begin{aligned} h(\tilde{\rho}(x, \Phi_\tau), t) &= h(\pi(x, t), \Phi_{\tau + \phi(x, t)}) \\ &= (\pi(x, t), \exp(2\pi i \tau) \exp(2\pi i \phi(x, t))) \\ &= \rho(h(x, \Phi_\tau), t) \end{aligned}$$

for $(x, \Phi_\tau) \in X \times S$ and $t \in R$. Hence ρ and $\tilde{\rho}$ are isomorphic. Since notions of minimality, distality and equicontinuity are invariant under isomorphisms, it is a distal minimal flow on $X \times S$ which is not equicontinuous. Furthermore, we have $\Lambda(\tilde{\rho}) = \Lambda(\pi)$ by Proposition 1.1.

3. SOME DISTAL FUNCTIONS

Proposition 3.1. *Let π be an equicontinuous minimal flow on a compact metric space X . Then*

(1) *If $\pi(x_0, t_n) \rightarrow x_0$ as $n \rightarrow \infty$ for some sequence $\{t_n\} \subset R$ and $x_0 \in X$, then we have $\pi(x, t_n) \rightarrow x$ as $n \rightarrow \infty$ for each $x \in X$.*

(2) *If $\pi(x_0, t_n) \rightarrow x_0$ and $\pi(x_0, s_n) \rightarrow x_0$ as $n \rightarrow \infty$ for some sequences $\{t_n\}, \{s_n\} \subset R$ and $x_0 \in X$, then $\pi(x_0, t_n + s_n) \rightarrow x_0$ as $n \rightarrow \infty$.*

Proof. Easy.

Proposition 3.2. *Let π be a distal minimal flow on a compact metric space X .*

(1) *For each $(x, y) \in X \times X$, there exists a sequence $\{t_n\} \subset R$ ($t_n \rightarrow \infty$ as $n \rightarrow \infty$) such that $(\pi(x, t_n), \pi(y, t_n)) \rightarrow (x, y)$ as $n \rightarrow \infty$.*

(2) *If F is a continuous function on X , then, for each $x \in X$, $f^x(t) = F(\pi(x, t))$ ($t \in R$) is a distal function, and $H(f^x) = \{F(\pi(y, \cdot)); y \in X\}$.*

Proof. (1) Since the product flow $\pi \times \pi$ on $X \times X$ defined by $\pi \times \pi((x, y), t) = (\pi(x, t), \pi(y, t))$ for $(x, y) \in X \times X$ and $t \in R$ is distal ([4], p. 37), it follows that $O_{\pi \times \pi}((x, y))$ is a minimal set of $\pi \times \pi$ ([4], p. 37). Hence (1) holds.

(2) Let h be a mapping from X into $C(R, C)$ defined by $h(x) = F(\pi(x, \cdot))$. Then h is a homomorphism from π to η . Hence, for each $x \in X$, $f^x(t) = F(\pi(x, t))$ for $t \in R$ is a distal function ([4], p. 37). The second assertion is obvious.

Proposition 3.3. *Let f be an almost periodic function on R . Then $\Lambda(\eta_f) = \tilde{\Lambda}_f$, where*

$$\Lambda_f = \left\{ \lambda \in R; \lim_{t \rightarrow \infty} \int_0^t f(s) \exp(-2\pi i \lambda s) ds \neq 0 \right\}$$

and $\tilde{\Lambda}_f$ is the least additive subgroup of R containing Λ_f .

Proof. See [2].

Lemma 3.4. *Let π be an equicontinuous minimal flow on a compact metric space X , let $\phi : X \times R \rightarrow R$ satisfy $(*)$ which is not uniformly continuous on $X \times R$, and let Φ be a continuous periodic function on R with the prime period 1. If $\Phi_\tau(\phi(x, t)) = \Phi_{\tau'}(\phi(x, t))$ for $x, y \in X$, $\tau, \tau' \in [0, 1)$ and $t \in R$, then we have $\Phi_\tau(n\phi(x, t)) = \Phi_{\tau'}(n\phi(y, t))$ for each natural number n and $t \in R$.*

Proof. Let $t \in R$. Since a flow ρ on $X \times K$ defined by

$$\rho((x, \xi), t) = (\pi(x, t), \xi \exp(2\pi i \phi(x, t)))$$

for $(x, \xi) \in X \times K$ and $t \in R$ is a distal minimal flow on $X \times K$, there exists a sequence $\{t_n\} \subset R$ ($t_n \rightarrow \infty$ as $n \rightarrow \infty$) such that $\pi(\pi(x, t), t_n) \rightarrow \pi(x, t)$, $\pi(\pi(y, t), t_n) \rightarrow \pi(y, t)$, $\phi(\pi(x, t), t_n) \rightarrow \phi(x, t) \pmod{1}$ and $\phi(\pi(y, t), t_n) \rightarrow \phi(y, t) \pmod{1}$ as $n \rightarrow \infty$ by Proposition 3.2. We shall show the lemma by mathematical induction. Since $\Phi_\tau(\phi(x, t) + \phi(\pi(x, t), t_n)) = \Phi_\tau(\phi(x, t + t_n)) = \Phi_{\tau'}(\phi(y, t + t_n)) = \Phi_{\tau'}(\phi(y, t) + \phi(\pi(y, t), t_n))$ for each n , we have $\Phi_\tau(2\phi(x, t)) = \Phi_{\tau'}(2\phi(y, t))$. Assume that

$$\phi(x, t + t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \rightarrow m\phi(x, t) \pmod{1}$$

and

$$\phi(y, t + t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \rightarrow m\phi(y, t) \pmod{1}$$

as $k \rightarrow \infty$ for $m \geq 2$, where $\{t_k^i\}$ are subsequences of $\{t_n\}$ for $1 \leq i \leq m-1$. Then $\Phi_\tau(m\phi(x, t)) = \Phi_{\tau'}(m\phi(y, t))$. Since $\pi(\pi(x, t), t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \rightarrow \pi(x, t)$ and $\pi(\pi(y, t), t_k^1 + t_k^2 + \cdots + t_k^{m-1}) \rightarrow \pi(y, t)$ as $n \rightarrow \infty$ by Proposition 3.1, we can choose a subsequence $\{k_n\}$ of natural numbers so that

$$\phi(\pi(\pi(x, t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n) \rightarrow \phi(x, t) \pmod{1}$$

and

$$\phi(\pi(\pi(y, t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n) \rightarrow \phi(y, t) \pmod{1}$$

as $n \rightarrow \infty$. For this sequence $\{k_n\}$ we have

$$\begin{aligned} & \Phi_\tau(\phi(x, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}) + \phi(\pi(\pi(x, t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)) \\ &= \Phi_\tau(\phi(x, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1} + t_n)) \\ &= \Phi_{\tau'}(\phi(y, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1} + t_n)) \\ &= \Phi_{\tau'}(\phi(y, t + t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}) + \phi(\pi(\pi(y, t), t_{k_n}^1 + t_{k_n}^2 + \cdots + t_{k_n}^{m-1}), t_n)). \end{aligned}$$

Hence we have $\Phi_\tau((m+1)\phi(x, t)) = \Phi_{\tau'}((m+1)\phi(y, t))$. Since $t \in R$ is arbitrary, we obtain the relation $\Phi_\tau(n\phi(x, t)) = \Phi_{\tau'}(n\phi(y, t))$ for each natural number n and $t \in R$.

Lemma 3.5. *Let $\alpha, \beta \in \mathbb{R}$. If α is an irrational number, and if $\beta \neq \frac{k}{m} - \frac{l}{m}\alpha$ ($k, l, m \in \mathbb{Z}, m \neq 0$), then, for each $(x, y) \in [0, 1) \times [0, 1)$, there exists a sequence $\{n_i\} \subset \mathbb{Z}$ ($n_i > 0$) such that $n_i\alpha \rightarrow x \pmod{1}$ and $n_i\beta \rightarrow y \pmod{1}$ as $i \rightarrow \infty$.*

Proof. Consider a flow π on $T^2 = K \times K$ defined by

$$\pi((\xi_1, \xi_2), t) = (\xi_1 e^{2\pi i \alpha t}, \xi_2 e^{2\pi i \beta t})$$

for $(\xi_1, \xi_2) \in T^2$ and $t \in \mathbb{R}$. Then, since α and β are rationally independent by the assumption, π is a minimal flow on T^2 . We can easily verify that $\Lambda(\pi) = \{l\alpha + m\beta; l, m \in \mathbb{Z}\}$ and that $1 \notin \{\frac{r}{k}; k \in \mathbb{Z} - \{0\}, r \in \Lambda(\pi)\}$. Hence the discrete flow π_1 defined by $\pi((\xi_1, \xi_2), 1)$ for $(\xi_1, \xi_2) \in T^2$ is minimal ([2]). Hence the orbit of π_1 is dense in T^2 . The lemma follows.

Theorem. *Let π be an equicontinuous minimal flow on a compact metric space X , ϕ a continuous function from $X \times \mathbb{R}$ to $\mathbb{R}$, and Φ a continuous periodic function with the prime period 1. If ϕ and Φ satisfy the following conditions:*

- (1) $\phi(x, t + s) = \phi(x, t) + \phi(\pi(x, t), s)$ for $x \in X$ and $t, s \in \mathbb{R}$,
- (2) $\lim_{t \rightarrow \infty} \frac{\phi(x, t)}{t} = 0$,
- (3) ϕ is not uniformly continuous on $X \times \mathbb{R}$ and $\phi(x, \cdot) \neq \phi(y, \cdot)$ for each pair of distinct points $x, y \in X$,
- (4) for each $\tau, \tau' \in [0, 1)$, there exists $t_{\tau\tau'} \in [0, 1)$ such that $\Phi_\tau(t_{\tau\tau'}) \neq \Phi_{\tau'}(-t_{\tau\tau'})$,

then $f(t) = \Phi(\phi(x_0, t))$ for $x_0 \in X$ and $t \in \mathbb{R}$ is a distal function, which is not almost periodic, and we have $\Lambda(\eta_f) = \Lambda(\pi)$.

Proof. It is enough to show η_f and $\tilde{\rho}$ in Proposition 2.4 are isomorphic. Define $h : X \times S \rightarrow \mathbb{R}$ by $h((x, \Phi_\tau)) = \Phi_\tau(0) = \Phi(\tau)$ for $(x, \Phi_\tau) \in X \times S$. Then h is continuous, $h(\tilde{\rho}((x, \Phi_\tau), t)) = \Phi(\tau + \phi(x, t))$ for $(x, \Phi_\tau) \in X \times S$ and $t \in \mathbb{R}$, and h is a homomorphism from $\tilde{\rho}$ to η_f . Assume that $h((x, \Phi_\tau)) = h((y, \Phi_{\tau'}))$, that is, $\Phi_\tau(\phi(x, t)) = \Phi_{\tau'}(\phi(y, t))$ for $(x, \Phi_\tau), (y, \Phi_{\tau'}) \in X \times S$ and $t \in \mathbb{R}$. If $x \neq y$, then $\phi(x, \cdot) \neq \phi(y, \cdot)$ by the assumption. Put $\delta(t) = \phi(x, t) - \phi(y, t)$ for $t \in \mathbb{R}$. Then, since $\delta(0) = 0$ and $\delta(t) \neq 0$, there exists t_1 such that $\delta(t_1)$ is irrational. Put $\alpha = \phi(x, t_1)$ and $\beta = \phi(y, t_1)$. We can assume without loss of generality that α is irrational. If $\beta = -\alpha + \frac{l}{k}$ ($k, l \in \mathbb{Z}, k \neq 0$), then $\Phi_\tau(nk\alpha) = \Phi_{\tau'}(-nk\alpha + nl) = \Phi_{\tau'}(-nk\alpha)$ for $n \in \mathbb{Z}$ ($n \geq 1$) by Lemma 3.4. Since $k\alpha$ is irrational, there exists a sequence $\{n_p\} \subset \mathbb{Z}$ ($n_p \geq 1$) such that $n_p k\alpha \rightarrow t \pmod{1}$ as $p \rightarrow \infty$ for each $t \in [0, 1)$. Hence we have $\Phi_\tau(t) \equiv \Phi_{\tau'}(-t)$. This is a contradiction to the assumption on Φ . Hence β is not of the form $-\alpha + \frac{l}{k}$. If $\beta = \frac{k}{m} - \frac{l}{m}\alpha$ ($k, m, l \in \mathbb{Z}, m \neq 0, |m| \neq |l|$), then we have $\Phi_\tau(nm\alpha) = \Phi_{\tau'}(kn - nl\alpha) = \Phi_{\tau'}(-nl\alpha)$ for $n \in \mathbb{Z}$ ($n \geq 1$) by Lemma 3.4. Hence we have $\Phi_\tau(mt) = \Phi_{\tau'}(-lt)$. If $|m| < |l|$, we obtain $\Phi_\tau(t) = \Phi_{\tau'}(-\frac{l}{m}t)$ for $t \in \mathbb{R}$, and $\Phi_\tau(t + \frac{m}{l}) = \Phi_{\tau'}(-\frac{l}{m}(t + \frac{m}{l})) = \Phi_{\tau'}(-\frac{l}{m}t - 1) = \Phi_{\tau'}(-\frac{l}{m}t) = \Phi_\tau(t)$. This implies Φ_τ has a period $|\frac{m}{l}| < 1$. This is a contradiction. If $\beta \neq \frac{k}{m} - \frac{l}{m}\alpha$ ($k, l, m \in \mathbb{Z}, m \neq 0$), then, by Lemma 3.5, for each $t \in [0, 1)$, there exists a sequence $\{n_p\} \subset \mathbb{Z}$ ($n_p \geq 1$) such that $n_p\alpha \rightarrow t \pmod{1}$ and $n_p\beta \rightarrow 0 \pmod{1}$ as $p \rightarrow \infty$. Hence we have $\Phi_\tau(t) \equiv \Phi_{\tau'}(0)$. This is a contradiction. The above implies that $\phi(x, t) = \phi(y, t)$ for $t \in \mathbb{R}$. Since $\phi(x, t)$ is unbounded, we have $\Phi_\tau = \Phi_{\tau'}$. These imply $x = y$ and $\tau = \tau' \pmod{1}$. Hence h is an injection, that is, h is an isomorphism from $\tilde{\rho}$ to η_f . It follows that f is a distal function, which is not almost periodic, and $\Lambda(\eta_f) = \Lambda(\pi)$.

Remark. A periodic function $\Phi(t) = \sin(2\pi t) + \sin(4\pi t)$ (for $t \in R$) satisfies the conditions in the Theorem.

Corollary. *Let $a(t)$ be an almost periodic function such that*

$$(1) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t a(s) ds = 0,$$

$$(2) \int_0^t a(s) ds \text{ is unbounded with respect to } t,$$

and let Φ be a continuous periodic function satisfying the conditions in the Theorem.

Then $f(t) = \Phi(\int_0^t a(s) ds)$ is a distal function, which is not almost periodic, and $\Lambda(\eta_f) = \tilde{\Lambda}_a$.

Proof. In the Theorem, put $\pi = \eta_a$ and $\phi(a^*, t) = \int_0^t a^*(s) ds$ for $a^* \in H(f)$ and $t \in R$. For the second assertion, use Proposition 3.3.

REFERENCES

1. J. Egawa, *Reparametrization and equicontinuous flows*, Proc. Japan Acad. **54** (1978), 202–205. MR **80a**:54075
2. ———, *Eigenvalues of compact minimal flows*, Math. Seminar Notes (Kobe University) **10** (1982), 281–291. MR **84f**:54054
3. ———, *Eigenvalues of harmonizable minimal flows*, Mathematica Japonica **31** (1986), 351–367. MR **87i**:54086
4. R. Ellis, *Lectures on topological dynamics*, W. A. Benjamin, 1969. MR **42**:2463
5. R. A. Johnson, *Almost periodic functions with unbounded integral*, Pacific J. of Math. **87** (1980), 347–362. MR **82e**:42013
6. E. Salehi, *Distal functions and unique ergodicity*, Trans. Amer. Math. Soc. **323** (1991), 703–713. MR **91e**:43009

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