A NOTE ON KAMENEV TYPE THEOREMS FOR SECOND ORDER MATRIX DIFFERENTIAL SYSTEMS

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ABSTRACT. Some oscillation criteria are given for the second order matrix differential system $Y'' + Q(t)Y = 0$, where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric, $t \in [t_0, \infty)$. These results improve oscillation criteria recently discovered by Erbe, Kong and Ruan by using a generalized Riccati transformation $V(t) = a(t)\{Y'(t)Y^{-1}(t) + f(t)I\}$, where I is the $n \times n$ identity matrix, $f \in C^1$ is a given function on $[t_0, \infty)$ and $a(t) = \exp\{-2 \int^t f(s) ds\}$.

Consider the second order linear differential system

$$(1) \quad Y'' + Q(t)Y = 0, \quad t \in [t_0, \infty),$$

where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric. A solution $Y(t)$ of (1) is said to be a nontrivial solution if $\det Y(t) \neq 0$ for at least one $t \in [t_0, \infty)$, and a nontrivial solution $Y(t)$ of (1) is said to be prepared if

$$(2) \quad Y^*(t)Y'(t) - (Y^*(t))'Y(t) \equiv 0, \quad t \in [t_0, \infty),$$

where for any matrix A , the transpose of A is denoted by A^* . System (1) is said to be oscillatory on $[t_0, \infty)$ in case the determinant of every nontrivial prepared solution vanishes on $[T, \infty)$ for each $T > t_0$.

For matrix system (1), many authors have given some important simple oscillation criteria (see [1], [2], [3], [6]). We particularly mention the results of Erbe, Kong and Ruan [3] who proved the following theorem.

Erbe, Kong and Ruan's Theorem. Let $H(t, s)$ and $h(t, s)$ be continuous on $D = \{(t, s) : t \geq s \geq t_0\}$ such that $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $\frac{\partial}{\partial s}H(t, s) = H_s(t, s)$ is nonpositive and continuous for $t \geq s \geq t_0$ and $h(t, s)$ is defined by

$$H_s(t, s) = -h(t, s)[H(t, s)]^{1/2}, \quad (t, s) \in D.$$

Finally, we assume that

$$(3) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t \left(H(t, s)Q(s) - \frac{1}{4}h^2(t, s)I \right) ds \right] = \infty,$$

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where $\lambda_1[A] \geq \lambda_2[A] \geq \cdots \geq \lambda_n[A]$ denotes the usual ordering of the eigenvalues of the symmetric matrix A , I is the $n \times n$ identity matrix. Then system (1) is oscillatory.

However, if $Q(t) = \text{diag}(\frac{\gamma}{t^2}, \frac{\alpha}{t^2})$ in (1), where $\gamma \geq \alpha > 0$ are constants, then

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{H(t, 1)} \lambda_1 \left[\int_1^t \left(H(t, s)Q(s) - \frac{1}{4}h^2(t, s)I \right) ds \right] \\ \leq \limsup_{t \rightarrow \infty} \int_1^t \frac{\gamma}{s^2} ds = \gamma < \infty, \end{aligned}$$

where $H(t, s), h(t, s)$ are defined as in Erbe, Kong and Ruan's theorem. Thus the above mentioned criteria of Erbe, Kong and Ruan cannot be applied to the Euler differential system

$$(4) \quad Y'' + \text{diag} \left(\frac{\gamma}{t^2}, \frac{\alpha}{t^2} \right) Y = 0,$$

where Y is a 2×2 matrix, $\gamma \geq \alpha > 0$ are constants. In fact, the Euler differential system (4) is oscillatory if $\gamma > \frac{1}{4}$ and nonoscillatory if $\gamma \leq \frac{1}{4}$.

The purpose of this note is to improve Erbe, Kong and Ruan's oscillation criteria by using a generalized Riccati transformation. Our main results are the following theorems.

Theorem 1. *Let $H(t, s)$ and $h(t, s)$ be continuous on $D = \{(t, s) : t \geq s \geq t_0\}$ such that $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ for $t > s \geq t_0$. We assume further that the partial derivative $\frac{\partial}{\partial s} H(t, s) = H_s(t, s)$ is nonpositive and is continuous for $t \geq s \geq t_0$ and $h(t, s)$ defined by*

$$H_s(t, s) = -h(t, s)[H(t, s)]^{1/2}, \quad (t, s) \in D.$$

If there exists a function $f \in C^1[t_0, \infty)$ such that

$$(5) \quad \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t \left(H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) ds \right] = \infty,$$

where $a(t) = \exp\{-2 \int_{t_0}^t f(s)ds\}$, $R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\}$, then equation (1) is oscillatory.

Proof. Suppose to the contrary that there exists a prepared solution $Y(t)$ of (1) which is not oscillatory. Without loss of generality, we may suppose that $\det Y(t) \neq 0$ for $t \geq t_0$. Define

$$V(t) = a(t)(Y'(t)Y^{-1}(t) + f(t)I), \quad t \geq t_0.$$

This and (1) imply

$$\begin{aligned} (6) \quad V'(t) &= -2f(t)V(t) + a(t)(Y''(t)Y^{-1}(t) - [Y'(t)Y^{-1}(t)]^2 + f'(t)I) \\ &= -\frac{1}{a(t)}V^2(t) - R(t), \quad t \geq t_0. \end{aligned}$$

Multiplying (6) with t replaced by s , by $H(t, s)$ and integrating from t_0 to t , we obtain

$$\begin{aligned}
 & \int_{t_0}^t H(t, s)R(s) ds \\
 &= - \int_{t_0}^t H(t, s)V'(s)ds - \int_{t_0}^t \frac{H(t, s)}{a(s)}V^2(s)ds \\
 &= -H(t, s)V(s)|_{t_0}^t - \int_{t_0}^t \left(-H_s(t, s)V(s) + \frac{H(t, s)}{a(s)}V^2(s) \right) ds \\
 &= H(t, t_0)V(t_0) - \int_{t_0}^t \left(-H_s(t, s)V(s) + \frac{H(t, s)}{a(s)}V^2(s) \right) ds \\
 &= H(t, t_0)V(t_0) - \int_{t_0}^t \left(h(t, s)\sqrt{H(t, s)}V(s) + \frac{H(t, s)}{a(s)}V^2(s) \right) ds \\
 &= H(t, t_0)V(t_0) + \frac{1}{4} \int_{t_0}^t a(s)h^2(t, s)I ds \\
 &\quad - \int_{t_0}^t \left[\sqrt{\frac{H(t, s)}{a(s)}}V(s) + \frac{1}{2}\sqrt{a(s)}h(t, s)I \right]^2 ds.
 \end{aligned}$$

Hence we have

$$\int_{t_0}^t \left(H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) ds \leq H(t, t_0)V(t_0), \quad t \geq t_0.$$

It follows that

$$(7) \quad \lambda_1 \left[\int_{t_0}^t \left(H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) ds \right] \leq \lambda_1[H(t, t_0)V(t_0)].$$

Since $H(t, t_0) > 0$ for $t > t_0$, dividing (7) by $H(t, t_0)$, we get

$$\begin{aligned}
 (8) \quad & \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t \left(H(t, s)R(s) - \frac{1}{4}a(s)h^2(t, s)I \right) ds \right] \\
 & \leq \frac{1}{H(t, t_0)} \lambda_1[H(t, t_0)V(t_0)] = \lambda_1[V(t_0)].
 \end{aligned}$$

Taking the upper limit in both sides of (8) as $t \rightarrow \infty$, the right-hand side is always bounded, which contradicts condition (5). This completes the proof.

Under a modification of the hypotheses of Theorem 1, we can obtain the following result.

Theorem 2. *In Theorem 1, if condition (5) is replaced by the conditions*

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t a(s)h^2(t, s) ds < \infty$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[\int_{t_0}^t H(t, s)R(s) ds \right] = \infty,$$

then system (1) is oscillatory.

If $f(t) = 0$, then Theorems 1 and 2 reduce to the Erbe, Kong and Ruan criterion [3].

Let $H(t, s) = (t - s)^\alpha$, $t \geq s \geq t_0$, where $\alpha > 1$ is an integer. By Theorem 1, we have the following result.

Corollary 1. *Let $\alpha > 1$ be an integer, suppose that there exists a function $f \in C^1[t_0, \infty)$ satisfying*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \lambda_1 \left[\int_{t_0}^t \left((t-s)^\alpha R(s) - \frac{\alpha^2}{4} (t-s)^{\alpha-2} a(s) I \right) ds \right] = \infty,$$

where $a(t) = \exp\{-2 \int^t f(s) ds\}$, $R(t) = a(t)\{Q(t) + f^2(t)I - f'(t)I\}$. Then Eq. (1) is oscillatory.

If $Q(t) = q(t)$, a scalar function, take $f(t) = 0$; then Corollary 1 reduces to the Kamenev criterion [5].

Example. Consider the Euler differential system (4) for $\gamma > \frac{1}{4}$ and let $f(t) = -\frac{1}{2t}$, then $a(t) = t$ and $f'(t) = \frac{1}{2t^2}$. This implies that

$$\int_1^t (t-s)^{\alpha-2} s ds = \left(\frac{t}{\alpha(\alpha-1)} + \frac{1}{\alpha} \right) (t-1)^{\alpha-1}.$$

Since $\alpha > 1$ is an integer, it follows from the book of Hardy, Littlewood and Pólya [4, Theorem 41], that

$$(t-s)^\alpha \geq t^\alpha - \alpha s t^{\alpha-1}, \quad \text{for } t \geq s \geq 1.$$

Then

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \lambda_1 \left[\int_1^t \left((t-s)^\alpha R(s) - \frac{\alpha^2}{4} (t-s)^{\alpha-2} a(s) I \right) ds \right] \\ &= \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \left((t-s)^\alpha s \left[\frac{\gamma}{s^2} + \frac{1}{4s^2} - \frac{1}{2s^2} \right] - \frac{\alpha^2}{4} (t-s)^{\alpha-2} s \right) ds \\ &= \frac{4\gamma-1}{4} \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \frac{(t-s)^\alpha}{s} ds - \frac{\alpha}{4(\alpha-1)} \\ &\geq \frac{4\gamma-1}{4} \limsup_{t \rightarrow \infty} \frac{1}{t^\alpha} \int_1^t \frac{t^\alpha - \alpha s t^{\alpha-1}}{s} ds - \frac{\alpha}{4(\alpha-1)} \\ &= \frac{4\gamma-1}{4} \limsup_{t \rightarrow \infty} \left[\log t - \frac{\alpha(t-1)}{t} \right] - \frac{\alpha}{4(\alpha-1)} = \infty. \end{aligned}$$

It follows from Corollary 1 that the Euler differential system (4) is oscillatory if $\gamma > \frac{1}{4}$. Now, let us consider the function defined by

$$H(t, s) = \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^\alpha, \quad \text{for } t \geq s \geq t_0,$$

where $\alpha > 1$ is an integer and θ is a positive continuous function on $[t_0, \infty)$ such that $\int_{t_0}^\infty (1/\theta(\tau)) d\tau = \infty$. Clearly

$$H(t, t) = 0 \quad \text{for } t \geq t_0, \quad \text{and} \quad H(t, s) > 0 \quad \text{for } t > s \geq t_0,$$

$$-\frac{\partial}{\partial s} H(t, s) = \frac{\alpha}{\theta(s)} \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^{\alpha-1},$$

and

$$h(t, s) = \frac{\alpha}{\theta(s)} \left(\int_s^t \frac{d\tau}{\theta(\tau)} \right)^{\alpha/2-1}, \quad t \geq s \geq t_0.$$

One of the important cases is to consider $\theta(\tau) = \tau^\beta$, where $\beta \leq 1$ is a real number. Here

$$\begin{aligned} H(t, s) &= \frac{1}{(1-\beta)^\alpha} [t^{1-\beta} - s^{1-\beta}]^\alpha, & \beta < 1, \\ &= \left(\log \frac{t}{s} \right)^\alpha, & \beta = 1, \end{aligned}$$

and

$$\begin{aligned} h(t, s) &= \frac{\alpha}{s^\beta} \left(\frac{t^{1-\beta} - s^{1-\beta}}{1-\beta} \right)^{\alpha/2-1}, & \beta < 1, \\ &= \frac{\alpha}{s} \left(\log \frac{t}{s} \right)^{\alpha/2-1}, & \beta = 1. \end{aligned}$$

Therefore, by applying Theorems 1 and 2 in the special case considered, we derive many new criteria for the oscillation of system (1)

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