ZERO DIVISORS AND $L^p(G)$

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(Communicated by J. Marshall Ash)

ABSTRACT. Let G be a discrete group, $\mathbb{C}G$ the group ring of G over $\mathbb{C}$ and $L^p(G)$ the Lebesgue space of G with respect to Haar measure. It is known that if G is torsion free elementary amenable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha * \beta \neq 0$. We will give a sufficient condition for this to be true when $p > 2$, and in the case $G = \mathbb{Z}^n$ we will give sufficient conditions for this to be false when $p > 2$.

1. INTRODUCTION

Let G be a discrete group, and let f be a complex valued function on G . We may represent f as a formal sum $\sum_{g \in G} a_g g$ where $a_g \in \mathbb{C}$ and $f(g) = a_g$. With respect to the counting measure on G the Lebesgue spaces $L^\infty(G)$, $C_0(G)$ and $L^p(G)$, $1 \leq p < \infty$, may be thought of in the following ways. $L^\infty(G)$ will consist of all formal sums where $\sup_{g \in G} |a_g| < \infty$, $C_0(G)$ will consist of all formal sums for which the set $\{g \mid |a_g| > \varepsilon\}$ is finite for all $\varepsilon > 0$, and $L^p(G)$ will consist of all formal sums where $\sum_{g \in G} |a_g|^p < \infty$. Let $\mathbb{C}G$ be the group ring of G over $\mathbb{C}$, so $\mathbb{C}G$ consists of all formal sums $\sum_{g \in G} a_g g$ where $a_g = 0$ for all but finitely many g . Then $\mathbb{C}G$ can also be thought of as the complex valued functions on G with compact support. The following inclusions are clear:

$$\mathbb{C}G \subseteq L^p(G) \subseteq C_0(G) \subseteq L^\infty(G).$$

For $\alpha = \sum_{g \in G} a_g g \in \mathbb{C}G$ and $\beta = \sum_{g \in G} b_g g \in L^p(G)$, $1 \leq p \leq \infty$, we define a multiplication $*$: $L^1(G) \times L^p(G) \rightarrow L^p(G)$ by

$$\alpha * \beta = \sum_{g, h} a_g b_h g h = \sum_{g \in G} \left(\sum_{h \in G} a_{gh^{-1}} b_h \right) g.$$

In [4] it is shown that if G is torsion free elementary amenable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha * \beta \neq 0$. A natural question to ask is does this remain true if 2 is replaced by any $p < \infty$.

Let α be an element of $L^1(G)$, and let $1 \leq p < \infty$. If there exists a nonzero β in $L^p(G)$ such that $\alpha * \beta = 0$, then we shall say that α is a *p-zero divisor*. If $\alpha * \beta \neq 0$ for all $\beta \in C_0(G) \setminus 0$, then we shall say that α is a *uniform nonzero divisor*. We shall see that if $\alpha \in \mathbb{C}\mathbb{Z}$, then α is a uniform nonzero divisor; however, this is not

Received by the editors November 29, 1994 and, in revised form, July 15, 1996.

1991 *Mathematics Subject Classification*. Primary 43A15; Secondary 43A25, 42B99.

Key words and phrases. Group ring, *p*-zero divisor, uniform nonzero divisor, Fourier transform, regular point, manifold of finite type.

true in general. In this paper we will give sufficient conditions for $\alpha \in \mathbb{C}\mathbb{Z}^n$, $n \geq 2$, to be a p -zero divisor. Also sufficient conditions will be given for α to be a uniform nonzero divisor. The proofs of these results will reveal a connection between p -zero divisors and the concepts of sets of uniqueness and spectral synthesis from Fourier analysis. Examples of p -zero divisors will also be given, along with an application to the theory of sets of uniqueness.

This work is part of the author's Ph.D. thesis done at Virginia Polytechnic Institute and State University, written under the supervision of Professor Peter Linnell, whose support and insightful suggestions were greatly appreciated.

2. STATEMENT OF MAIN RESULTS

For the rest of this paper, unless otherwise stated, assume that G is abelian. Let Γ be the group of homomorphisms from G to the unit circle in $\mathbb{C}$. Γ is known as the dual group of G . Also a topology can be defined on Γ that makes it a topological group. If $f = \sum_{g \in G} a_g g \in L^1(G)$, the Fourier transform of f is defined by

$$\hat{f}(\gamma) = \sum_{g \in G} a_g(g^{-1}, \gamma)$$

where $\gamma \in \Gamma$ and $(g^{-1}, \gamma) = \gamma(g^{-1})$. Let $Z(f)$ denote the elements γ in Γ such that $\hat{f}(\gamma) = 0$. We shall prove

Theorem 1. *Let G be torsion free and let $\alpha \in L^1(G)$. If $Z(\alpha)$ is a proper subgroup of Γ , then α is a uniform nonzero divisor.*

It is well known that $\mathbb{T}^n$, where $\mathbb{T} = [-\pi, \pi]/\{-\pi \sim \pi\}$, is the dual group of $\mathbb{Z}^n$. Let $V = (-\pi, \pi)^n$ and identify V with an open subspace of $\mathbb{T}^n$. Using Theorem 1 we will then show that if $\alpha \in L^1(\mathbb{Z}^n)$ and $Z(\alpha)$ is contained in a finite union of proper closed cosets on $\mathbb{T}^n$, then α is a uniform nonzero divisor. Let $\alpha \in L^1(\mathbb{Z}^n)$ and $x_0 \in E := Z(\alpha) \cap V$. We shall say that x_0 is a *regular point* if there exists an open neighborhood U of x_0 such that $F_{x_0} = U \cap E$ is a smooth m -dimensional submanifold of V , where m is a natural number. Consider F_{x_0} in a sufficiently small neighborhood of x_0 and write F_{x_0} as the image of a smooth mapping $\phi : W \rightarrow V$, where W is a neighborhood of the origin in $\mathbb{R}^m$. Also assume that the vectors $\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_m}$ are linearly independent for each $x \in W$. Now suppose for each $y_0 \in W$ and each unit vector $\eta \in V$, there is a multi-index j , with $|j| \geq 1$, so that

$$\partial_x^j [\phi(x) \cdot \eta]|_{x=y_0} \neq 0$$

(where of course $\phi(x) \cdot \eta$ is the usual Euclidean inner product). The smallest k so that, for each unit vector η , there exists a j with $|j| \leq k$ for which

$$\partial_x^j [\phi(x) \cdot \eta]|_{x=y_0} \neq 0,$$

is called the *type* of ϕ (and the type of F_{x_0}) at y_0 . Also, if $W_1 \subset W$ is a compact set, the type of ϕ in W_1 is defined to be the maximum of the type of the $y_0 \in W_1$. We shall prove

Theorem 2. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$, and suppose that x_0 is a regular point in $Z(\alpha)$. If F_{x_0} is of finite type k , then α is a p -zero divisor for $p > nk$.*

3. PRELIMINARIES

Let p be a real number that is greater than or equal to one, and let q denote the conjugate index of p , i.e., if $p > 1$ then $\frac{1}{p} + \frac{1}{q} = 1$, and $q = \infty$ if $p = 1$. Let $y \in G$, let $f = \sum_{g \in G} a_g g \in L^p(G)$, and denote the right translation of f by y with f_y , where $f_y(x) = f(xy^{-1})$. Let $h = \sum_{g \in G} b_g g \in L^q(G)$ and define a map $\langle \cdot, \cdot \rangle : L^p(G) \times L^q(G) \rightarrow \mathbb{C}$ by $\langle f, h \rangle = \sum_{g \in G} a_g \overline{b_g}$. Fix $h \in L^q(G)$. Then $\langle \cdot, h \rangle$ is a continuous linear functional on $L^p(G)$. By the Riesz representation theorem every continuous linear functional on $L^p(G)$ is of this form. Let $T^p[f]$ be the closure in $L^p(G)$ of the set of linear combinations of translates of f . By the Hahn-Banach theorem, $T^p[f] = L^p(G)$ if and only if no nonzero continuous linear functional on $L^p(G)$ vanishes on all translates of f . For $\beta = \sum_{g \in G} a_g g \in L^p(G)$, set $\beta^* = \sum_{g \in G} \overline{a_g} g^{-1}$.

Lemma 1. *Let $\alpha = \sum_{g \in G} a_g g$ be an element of $L^1(G)$. Then α is a p -zero divisor if and only if $T^q[\alpha]$ is not equal to $L^q(G)$.*

Proof. Let $\beta = \sum_{g \in G} b_g g \in L^p(G)$; then

$$\langle \alpha_y, \beta^* \rangle = \sum_{g \in G} a_{gy^{-1}} \overline{b_g} = \sum_{g \in G} a_{y^{-1}g} \overline{b_g} = (\alpha * \beta^*)(y^{-1}).$$

So $\alpha * \beta^* = 0$ if and only if $\langle \alpha_y, \beta^* \rangle = 0$ for all $y \in G$. □

Remark. If $\alpha \in \mathbb{C}G$, then α is a uniform nonzero divisor if and only if $T^q[\alpha] = L^q(G)$ for all q , $1 < q < \infty$. If $Z(\alpha)$ is finite, then α is a uniform nonzero divisor by Theorem 4.2 of [2]. Hence, nonzero elements of $\mathbb{C}Z$ are uniform nonzero divisors.

Recall that Γ denotes the dual group of G . Let $M(\Gamma)$ be the set of bounded regular Borel measures on Γ . Let E be a closed subset of Γ . Denote by $M(E)$ the elements in $M(\Gamma)$ that are concentrated on E . For $\mu \in M(\Gamma)$, the Fourier-Stieltjes transform of μ is defined by

$$\hat{\mu}(x) = \int_{\Gamma} (x, \gamma) d\mu(\gamma)$$

where $x \in G$. Clearly $\hat{\mu} \in L^\infty(G)$, so $\hat{\mu}$ is a continuous linear functional on $L^1(G)$. Set $\hat{\mu}^*(g) = \overline{\hat{\mu}(g^{-1})}$.

Lemma 2. *If $f = \sum_{g \in G} a_g g \in L^1(G)$ and $\mu \in M(Z(f))$, then $\langle f_y, \hat{\mu}^* \rangle = 0$ for all $y \in G$.*

Proof.

$$\begin{aligned} \langle f, \hat{\mu}^* \rangle &= \sum_{g \in G} a_g \overline{\hat{\mu}^*(g)} = \sum_{g \in G} a_g \int_{\Gamma} (g^{-1}, \gamma) d\mu(\gamma) \\ &= \int_{\Gamma} \sum_{g \in G} a_g (g^{-1}, \gamma) d\mu(\gamma) = 0 \end{aligned}$$

since $\sum_{g \in G} a_g (g^{-1}, \gamma) = \hat{f}(\gamma)$. The lemma now follows after we make the observation that $Z(f_y) = Z(f)$. □

Remark. If $\alpha \in L^1(G)$ and $\mu \in M(Z(\alpha))$, then $\alpha * \hat{\mu}^* = 0$.

Let E be a closed subset of Γ , $I(E)$ the set of all $f \in L^1(G)$ such that $E \subseteq Z(f)$ and $j(E)$ the set of all $f \in L^1(G)$ such that $E \subseteq O \subseteq Z(f)$, where O is an open set in Γ . Denote by $J(E)$ the closure of $j(E)$ in the $L^1(G)$ -norm. $I(E)$ and $J(E)$ are translation-invariant subspaces of $L^1(G)$ and $J(E) \subseteq I(E)$. Let $\Phi(E) = \{h \in L^\infty(G) | \langle f, h \rangle = 0, f \in I(E)\}$ and $\Psi(E) = \{h \in L^\infty(G) | \langle f, h \rangle = 0, f \in J(E)\}$. The following are easily deduced from Chapter 7 in [6], and will be used in the sequel.

1. $\Phi(E)$ is the weak*-closed subspace of $L^\infty(G)$ generated by E .
2. $\Phi(E)$ is the weak*-closure of $\{\hat{\mu} | \mu \in M(E)\}$.
3. If $\alpha * h = 0$, where $\alpha \in \mathbb{C}G$ and $h \in L^\infty(G)$, then $h \in \Psi(Z(\alpha))$.

E is said to be a set of *spectral synthesis* (S -set) if $I(E) = J(E)$. E is a set of *uniqueness* if $\Psi(E) \cap C_0(G) = 0$. If E is not a set of uniqueness, then E is a set of *multiplicity*.

4. A RESULT ON SETS OF UNIQUENESS

In this section we will prove a result on the union of sets of uniqueness. This result will be used to show certain elements of $L^1(\mathbb{Z}^n)$ are uniform nonzero divisors. Our result may or may not be new, but we record it here for completeness. For more information about sets of uniqueness see [1].

We will begin by showing that given a closed set E on $\mathbb{T}^n$, there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. Before we do this we need to prove a technical lemma. If $f \in L^1(\mathbb{Z}^n)$, the Fourier transform of f is

$$\hat{f}(t) = \sum_{m \in \mathbb{Z}^n} f(m) e^{-i(m \cdot t)}$$

where $t \in \mathbb{T}^n$ and $m \cdot t$ is the usual Euclidean inner product. The Fourier transform induces an isometry between $L^2(\mathbb{Z}^n)$ and $L^2(\mathbb{T}^n)$. For $g \in L^2(\mathbb{T}^n)$, the inverse map to the Fourier transform is given by

$$\check{g}(m) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} g(t_1, \dots, t_n) e^{i(\sum_{k=1}^n m_k t_k)} dt_1 \cdots dt_n$$

where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Recall that $C^\infty(\mathbb{T}^n)$ denotes the infinitely differentiable functions on $\mathbb{T}^n$. The next lemma is a generalization of exercise 4 from page 30 of [3].

Lemma 3. *If $g \in C^\infty(\mathbb{T}^n)$, then*

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} |\check{g}(m)| &\leq \|g\|_{L^1(\mathbb{T}^n)} + \sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^n} \left\| \frac{\partial^n g}{\partial x_1 \cdots \partial x_n} \right\|_{L^2(\mathbb{T}^n)} \\ &\quad + \sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^{n-1}} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} \left\| \frac{\partial^{n-1} g}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \right\|_{L^2(\mathbb{T}^n)} \\ &\quad + \sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right)^{n-2}} \sum_{1 \leq i_1 < \cdots < i_{n-2} \leq n} \left\| \frac{\partial^{n-2} g}{\partial x_{i_1} \cdots \partial x_{i_{n-2}}} \right\|_{L^2(\mathbb{T}^n)} \\ &\quad + \cdots + \sqrt{\sum_{k=1}^{\infty} \frac{2}{k^2}} \sum_{1 \leq i \leq n} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^2(\mathbb{T}^n)}. \end{aligned}$$

Proof. Since our proof is easy to generalize, we will prove the lemma for the case $n = 2$. To begin with

$$\begin{aligned} \sum_{(n_1, n_2) \in \mathbb{Z}^2} |\check{g}(n_1, n_2)| &= |\check{g}(0, 0)| + \sum_{n_1 \neq 0} |\check{g}(n_1, 0)| \\ &\quad + \sum_{n_2 \neq 0} |\check{g}(0, n_2)| + \sum_{n_1 \neq 0, n_2 \neq 0} |\check{g}(n_1, n_2)|. \end{aligned}$$

Using integration by parts we obtain

$$\begin{aligned} \frac{\partial^2 g}{\partial x_1 \partial x_2}(n_1, n_2) &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\partial^2 g}{\partial x_1 \partial x_2}(x_1, x_2) e^{i(n_1 x_1 + n_2 x_2)} dx_1 dx_2 \\ &= \frac{-n_1 n_2}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g(x_1, x_2) e^{i(n_1 x_1 + n_2 x_2)} dx_1 dx_2 \\ &= -n_1 n_2 \check{g}(n_1, n_2). \end{aligned}$$

With the help of the Cauchy-Schwarz inequality and Parseval's relation we obtain

$$\begin{aligned} \sum_{n_1 \neq 0, n_2 \neq 0} |\check{g}(n_1, n_2)| &= \sum_{n_1 \neq 0, n_2 \neq 0} \left| \frac{1}{n_1 n_2} \frac{\partial^2 g}{\partial x_1 \partial x_2}(n_1, n_2) \right| \\ &\leq \sqrt{\left(\sum_{k=1}^{\infty} \frac{2}{k^2} \right)^2} \left\| \frac{\partial^2 g}{\partial x_1 \partial x_2} \right\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

It can be shown by similar calculations that

$$\sum_{n_1 \neq 0} |\check{g}(n_1, 0)| \leq \sqrt{\sum_{k=1}^{\infty} \left(\frac{2}{k^2} \right)^2} \left\| \frac{\partial g}{\partial x_1} \right\|_{L^2(\mathbb{T}^2)}$$

and

$$\sum_{n_2 \neq 0} |\check{g}(0, n_2)| \leq \sqrt{\sum_{k=1}^{\infty} \left(\frac{2}{k^2} \right)^2} \left\| \frac{\partial g}{\partial x_2} \right\|_{L^2(\mathbb{T}^2)}.$$

The lemma follows after we make the observation that $|\check{g}(0, 0)| \leq \|g\|_{L^1(\mathbb{T}^2)}$. $\square$

Corollary 1. *If $g \in C^\infty(\mathbb{T}^n)$, then there exists an $f \in L^1(\mathbb{Z}^n)$ such that $\hat{f} = g$.*

Let E be a closed subset of $V = (-\pi, \pi)^n$ and set d equal to the Euclidean distance from E to $\mathbb{R}^n \setminus V$. Let U_r be the set of points of $\mathbb{R}^n$ at a distance less than $\frac{d}{r}$ from E , where r is a natural number. Edwards constructs a function F on $\mathbb{R}^n$ [2, pp. 229–230] such that $Z(F) = E$ and $F \in C^\infty(\mathbb{R}^n)$, the infinitely differentiable functions on $\mathbb{R}^n$. By using the U_r 's defined above in place of the U_r 's used in Edward's argument, we construct a function g such that $Z(g) = E$ and g is constant on the frontier of V , so $g \in C^\infty(\mathbb{T}^n)$. By the above corollary we see that given a closed set $E \subseteq V$ there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. Now suppose that E is any closed set on $\mathbb{T}^n$. Write E as the union of two nonempty closed sets E_1, E_2 and let $f_1, f_2 \in L^1(\mathbb{Z}^n)$ such that $Z(f_i) = E_i$. Since $L^1(\mathbb{Z}^n)$ is a ring and $Z(f_1 * f_2) = Z(f_1) \cup Z(f_2) = E$, we can conclude that given a closed set E on $\mathbb{T}^n$ there exists an $f \in L^1(\mathbb{Z}^n)$ such that $Z(f) = E$. We are now ready to give our result.

Proposition 1. *Suppose that E_1, E_2 are closed sets of uniqueness on $\mathbb{T}^n$. If $E_1 \cup E_2$ is an S -set, then $E_1 \cup E_2$ is a set of uniqueness.*

Proof. From the discussion above we know that there exist functions f_1, f_2 in $L^1(\mathbb{Z}^n)$ such that $E_i = Z(f_i)$. Let $h \in C_0(\mathbb{Z}^n)$. Since the E_i 's are sets of uniqueness and $f_1 * f_2 \in L^1(\mathbb{Z}^n)$, it follows that $(f_1 * f_2) * h = 0$ if and only if $h = 0$. Since $E_1 \cup E_2 = Z(f_1 * f_2)$ and $E_1 \cup E_2$ is an S -set we see that $E_1 \cup E_2$ is a set of uniqueness. $\square$

5. PROOF OF THEOREM 1 AND RELATED RESULTS

Let $A = \{a \in G \mid \gamma(a) = 1 \text{ for all } \gamma \in Z(\alpha)\}$. A is known as the annihilator subgroup of $Z(\alpha)$ and $A \neq 0$ since $Z(\alpha) \neq \Gamma$. Since G is torsion free, A is infinite. Let $h \in \Phi(Z(\alpha))$ and fix $g \in G$ such that $h(g) \neq 0$; now $|h(ga)| = |h(g)|$ for all $a \in A$ since $\gamma(ga) = \gamma(g)\gamma(a) = \gamma(g)$. Thus h is not in $C_0(G)$. $Z(\alpha)$ is an S -set (Theorem 7.5.2(d), [6]), so $\Psi(Z(\alpha)) \cap C_0(G) = 0$ and the theorem follows.

Note that the theorem is still true if the subgroup generated by $Z(\alpha)$ is not all of Γ . The following corollary is undoubtedly well known. However, we cannot find a suitable reference, so we record it here for completeness.

Corollary 2. *If $Z(\alpha)$ is a proper subgroup of Γ , then $Z(\alpha)$ is a set of uniqueness.*

Corollary 3. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$. If $Z(\alpha)$ is contained in a finite union of proper closed cosets, then α is a uniform nonzero divisor.*

Proof. Let $\beta(x_1, \dots, x_n) = x_1 - x_n$, so $\hat{\beta}(\theta_1, \dots, \theta_n) = e^{-i\theta_1} - e^{-i\theta_n}$ and $Z(\beta) = \{(\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_1) \mid -\pi \leq \theta_k \leq \pi\}$. $Z(\beta)$ is a proper subgroup of $\mathbb{T}^n$, so is a set of uniqueness (note that β is a uniform nonzero divisor). Assume for now that $Z(\alpha)$ is a coset in $\mathbb{T}^n$. By translation we may assume that $Z(\alpha) \subseteq Z(\beta)$, so $\Psi(Z(\alpha)) \subseteq \Psi(Z(\beta))$, thus $Z(\alpha)$ is a set of uniqueness and α is a uniform nonzero divisor. The corollary now follows from Theorem 7.5.2 in [6] and Proposition 1. $\square$

6. PROOF OF THEOREM 2 AND RELATED RESULTS

Let x_0 be a regular point of $Z(\alpha)$, such that F_{x_0} is a submanifold of finite type k . Let μ be a smooth nonzero mass density on the closure of F_{x_0} in $\mathbb{R}^n$, so $\mu \in M(Z(\alpha))$ and $\alpha * \hat{\mu}^* = 0$. The theorem will be proved once we show that $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. Let $0 \neq \eta \in \mathbb{Z}^n$; it is shown in Chapter 8 of [7] that

$$|\hat{\mu}(\eta)| \leq C|\eta|^{-\frac{1}{k}}$$

where $|\cdot|$ is the usual Euclidean norm and C is some constant. Set $\eta = (m_1, \dots, m_n)$, $\eta' = (m_1 + 1, \dots, m_n + 1)$ and let j be the least integer greater than $n^{\frac{1}{2}}$. The triangle inequality implies that $|\eta'| - n^{\frac{1}{2}} \leq |\eta|$, so for $|\eta| > n^{\frac{1}{2}} - 1$ we obtain

$$\begin{aligned} |\hat{\mu}(\eta)|^p &\leq (C(|\eta|)^{-\frac{1}{k}})^p \\ &\leq (C(|\eta'| - n^{\frac{1}{2}})^{-\frac{1}{k}})^p \\ &\leq \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p dx_1 \cdots dx_n \end{aligned}$$

where $f(x) = ((x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} - n^{\frac{1}{2}})^{-\frac{1}{k}}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $N \in \mathbb{N}$ we have

$$(1) \quad \sum_{\eta \in \mathbb{Z}^n, |\eta| \leq N} |\hat{\mu}(\eta)|^p \leq \sum_{|\eta| < n^{\frac{1}{2}} + 1} |\hat{\mu}(\eta)|^p + \sum_{1+n^{\frac{1}{2}} \leq |\eta| \leq N+j} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (Cf(x))^p dx_1 \cdots dx_n.$$

Let p be a real number strictly greater than nk ; so

$$\int_{|x| \geq 1+n^{\frac{1}{2}}} (f(x))^p dx_1 \cdots dx_n$$

is finite, hence

$$\sum_{\eta \in \mathbb{Z}^n, |\eta| \geq 1+n^{\frac{1}{2}}} \int_{m_n}^{m_n+1} \cdots \int_{m_1}^{m_1+1} (f(x))^p dx_1 \cdots dx_n$$

converges. Letting $N \rightarrow \infty$ in (1) we obtain $\sum_{\eta \in \mathbb{Z}^n} |\hat{\mu}(\eta)|^p < \infty$; therefore, $\hat{\mu}^* \in L^p(\mathbb{Z}^n)$ for $p > nk$. The proof is complete.

Corollary 4. *If α is as in Theorem 2, then $T^q[\alpha]$ does not equal $L^q(\mathbb{Z}^n)$ for q such that $1 \leq q < \frac{nk}{nk-1}$.*

Proof. Use Lemma 1. □

Corollary 5. *Let $\alpha \in L^1(\mathbb{Z}^n)$, $n \geq 2$, and suppose that there exists an $x_0 \in Z(\alpha)$ such that F_{x_0} is an $n-1$ dimensional submanifold of V . If F_{x_0} has strictly positive Gaussian curvature, then α is a p -zero divisor for $p > \frac{2n}{n-1}$.*

Proof. Let $0 \neq \eta \in \mathbb{Z}^n$; then by [5]

$$|\hat{\mu}(\eta)| \leq C|\eta|^{-\frac{n-1}{2}}.$$

Now proceed as in the theorem. □

7. EXAMPLES

In [4] it is shown that if G is torsion free elementary amenable, $0 \neq \alpha \in \mathbb{C}G$, then α is not a p -zero divisor for $p \leq 2$. We will give an example to show that this cannot be improved. Let $p > 2$ be given and pick an integer n such that $2 < \frac{2n}{n-1} < p$. Let

$$\alpha(x_1, \dots, x_n) = \frac{2n-1}{2} - \frac{1}{2} \left(\sum_{k=1}^n (x_k + x_k^{-1}) \right),$$

so $\alpha(x_1, \dots, x_n) \in \mathbb{C}\mathbb{Z}^n$, $\hat{\alpha}(t_1, \dots, t_n) = \frac{2n-1}{2} - \sum_{k=1}^n \cos t_k$. Near $(0, \dots, 0, \frac{\pi}{3})$, $Z(\alpha)$ is of the form $\{(t, g(t)) | t \in U\}$, where U is a bounded open set containing 0 in $\mathbb{R}^{n-1}$, $t = (t_1, \dots, t_{n-1})$ and $g(t) = \cos^{-1}(\frac{2n-1}{2} - \sum_{k=1}^{n-1} \cos t_k)$. A computation shows that the rank of the matrix $(\frac{\partial^2 g(t)}{\partial t_i \partial t_k})$ is $n-1$ at $t=0$, hence $Z(\alpha)$ has strictly positive curvature. Therefore, $\alpha(x_1, \dots, x_n)$ is an r -zero divisor for $r > \frac{2n}{n-1}$.

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