MINKOWSKI'S INEQUALITY FOR TWO VARIABLE DIFFERENCE MEANS

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ABSTRACT. We study Minkowski's inequality

$$D_{a,b}(x_1 + x_2, y_1 + y_2) \leq D_{a,b}(x_1, y_1) + D_{a,b}(x_2, y_2) \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}_+)$$

and its reverse where $D_{a,b}$ is the difference mean introduced by Stolarsky. We give necessary and sufficient conditions (concerning the parameters a, b) for the inequality above (and for its reverse) to hold.

1. INTRODUCTION

Let $a, b \in \mathbb{R}$ be two distinct nonzero real numbers. Cauchy's mean value theorem, applied to the functions $t \rightarrow t^a, t \rightarrow t^b$ on a subinterval $[y, x]$ of the positive real numbers $\mathbb{R}_+$, gives that the number

$$\xi = \left(\frac{b(x^a - y^a)}{a(x^b - y^b)} \right)^{\frac{1}{a-b}}$$

lies between y and x . Thus $D_{a,b}(x, y) = \xi$ is a mean value of x and y . This mean value was first defined by Stolarsky [20] who showed that $D_{a,b}(x, y)$ can be extended continuously to the domain $R = \{ (a, b, x, y) \mid a, b \in \mathbb{R}, x, y \in \mathbb{R}_+ \}$. This extension is given by

$$(1) \quad D_{a,b}(x, y) = \begin{cases} \left(\frac{b(x^a - y^a)}{a(x^b - y^b)} \right)^{\frac{1}{a-b}} & \text{if } ab(a-b)(x-y) \neq 0, \\ \left(\frac{x^a - y^a}{a(\ln x - \ln y)} \right)^{\frac{1}{a}} & \text{if } a(x-y) \neq 0, b = 0, \\ \left(\frac{b(\ln x - \ln y)}{x^b - y^b} \right)^{-\frac{1}{b}} & \text{if } b(x-y) \neq 0, a = 0, \\ \exp \left(-\frac{1}{a} + \frac{x^a \ln x - y^a \ln y}{x^a - y^a} \right) & \text{if } b(x-y) \neq 0, a = b, \\ \sqrt{xy} & \text{if } x-y \neq 0, a = b = 0, \\ x & \text{if } x-y = 0. \end{cases}$$

$D_{a,b}(x, y)$ is called the difference mean or Stolarsky's mean of x and y . The power mean $M_a(x, y)$ with exponent a can be obtained as $D_{2/a,1}(x, y)$; thus $D_{2/1}, D_{0/0}$, and

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D_{-2-1} are the arithmetic, geometric, and harmonic means respectively. D_{01} and D_{11} are called logarithmic and identric means, respectively.

The comparison problem

$$(2) \quad D_{ab}(x, y) \leq D_{cd}(x, y)$$

for difference means on $\mathbb{R}_+$ (i.e. if in (2) $x, y \in \mathbb{R}_+$) was solved by Leach and Sholander [11]. Páles [15] gave a new proof for this result. In [16] Páles solved the comparison problem (2) on any subinterval (α, β) of $\mathbb{R}_+$. Several particular inequalities involving D_{ab} and its special cases were dealt with by Alzer [2], Brenner [3], Brenner and Carlson [4], Burk [5], Carlson [6], Dodd [7], Leach and Sholander [10], Lin [13], Pittinger [17], [18], Sándor [22], Seiffert [23], [24], Stolarsky [20], and Székely [25]. Neuman [14] studied multivariable weighted logarithmic means; Leach and Sholander [12] dealt with difference means with several variables.

Some of these means have applications in electrostatics [1], [19], in heat conductions and chemical problems [26]. Of particular recent interest is the occurrence of these means in signal processing theory in connection with time-frequency distributions. See [8], especially pp. 17–20, 34–36, 38.

The aim of this paper is to find necessary and sufficient conditions for Minkowski's inequality

$$(3) \quad D_{ab}(x_1 + x_2, y_1 + y_2) \leq D_{ab}(x_1, y_1) + D_{ab}(x_2, y_2) \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}_+)$$

(and its reverse inequality) to hold.

2. RESULTS

Our results can be formulated in three theorems.

Theorem 1. *The inequality (3) holds if and only if*

$$(4) \quad a + b \geq 3, \quad \min\{a, b\} \geq 1.$$

Theorem 2. *The inequality*

$$(5) \quad D_{ab}(x_1 + x_2, y_1 + y_2) \geq D_{ab}(x_1, y_1) + D_{ab}(x_2, y_2) \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}_+)$$

holds if and only if

$$(6) \quad a + b \leq 3, \quad \min\{a, b\} \leq 1.$$

Figure 1 shows the domain of points (a, b) satisfying condition (4) and (6).

The following numerical examples illustrate the typical inequalities that can be obtained from Theorem 1 and Theorem 2. (For the sake of simplicity, we assume that $(x_1 - y_1)(x_2 - y_2)(x_1 + x_2 - y_1 - y_2) \neq 0$ in the inequalities below.)

The points $(2, 2)$, $(3, 1)$, $(4, 2)$ satisfy (4). Thus, Theorem 1 yields the following inequalities (after multiplying by some constant factor):

$$\begin{aligned} \left(\frac{(x_1 + x_2)^{(x_1 + x_2)^2}}{(y_1 + y_2)^{(y_1 + y_2)^2}} \right)^{\frac{1}{(x_1 + x_2)^2 - (y_1 + y_2)^2}} &\leq \left(\frac{x_1^{(x_1^2)}}{y_1^{(y_1^2)}} \right)^{\frac{1}{x_1^2 - y_1^2}} + \left(\frac{x_2^{(x_2^2)}}{y_2^{(y_2^2)}} \right)^{\frac{1}{x_2^2 - y_2^2}}, \\ &\sqrt{(x_1 + x_2)^2 + (x_1 + x_2)(y_1 + y_2) + (y_1 + y_2)^2} \\ &\leq \sqrt{x_1^2 + x_1 y_1 + y_1^2} + \sqrt{x_2^2 + x_2 y_2 + y_2^2}, \\ \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} &\leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}. \end{aligned}$$

FIGURE 1

(The last inequality is equivalent to the two dimensional Cauchy-Schwarz inequality.) Analogously, the points $(0, 0)$, $(2, -1)$, $(1, 0)$ satisfy (6). Therefore, by Theorem 2, we obtain

$$\begin{aligned} \sqrt{(x_1 + x_2)(y_1 + y_2)} &\geq \sqrt{x_1 y_1} + \sqrt{x_2 y_2}, \\ \sqrt[3]{(x_1 + x_2)^2(y_1 + y_2) + (x_1 + x_2)(y_1 + y_2)^2} &\geq \sqrt[3]{x_1^2 y_1 + x_1 y_1^2} + \sqrt[3]{x_2^2 y_2 + x_2 y_2^2}, \\ \frac{x_1 + x_2 - y_1 - y_2}{\ln(x_1 + x_2) - \ln(y_1 + y_2)} &\geq \frac{x_1 - y_1}{\ln x_1 - \ln y_1} + \frac{x_2 - y_2}{\ln x_2 - \ln y_2}. \end{aligned}$$

Since the domains given by (4) and (6) have just two common points $(1, 2)$ and $(2, 1)$, we have the following

Corollary 1. *The identity*

$$(7) \quad D_{ab}(x_1 + x_2, y_1 + y_2) = D_{ab}(x_1, y_1) + D_{ab}(x_2, y_2) \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}_+)$$

holds if and only if

$$(a, b) = (1, 2) \quad \text{or} \quad (a, b) = (2, 1).$$

Concerning equality in (3) and (6), we have

Theorem 3. *Let $(a, b) \in \mathbb{R}^2$, $(a, b) \neq (1, 2)$, $(a, b) \neq (2, 1)$. Suppose that (3) (or (5)) holds. Then equality is valid in (3) (or in (5)) if and only if*

$$(8) \quad \frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

3. PROOFS

The properties of D_{ab} are summarized in

Lemma 1. *Let, for $(a, b, x, y) \in R$, $d(a, b, x, y) := D_{ab}(x, y)$. Then we have*

- (i) $d(a, b, x, y) = d(b, a, x, y) = d(a, b, y, x)$,
- (ii) $d(a, b, tx, ty) = td(a, b, x, y)$,
- (iii) $d \in C^\infty(R)$

for all $(a, b, x, y) \in R, t \in \mathbb{R}_+$.

Proof. Properties (i) and (ii) follow immediately from the definition of d and from (1).

To prove that d is infinitely many times differentiable in all of its variables we need an integral form of d . Let

$$(9) \quad F(a, x, y) := \ln \int_0^1 (ux + (1-u)y)^{a-1} du \quad (a \in \mathbb{R}; x, y \in \mathbb{R}_+).$$

Since

$$0 < \min\{x, y\} \leq ux + (1-u)y \leq \max\{x, y\},$$

the integrand in (9) is a positive function having continuous partial derivatives of any order with respect to a, x, y on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. Thus both F and f defined by

$$f(a, x, y) = \frac{\partial}{\partial a} F(a, x, y)$$

are in $C^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+)$.

It is easy to check that for all $x, y \in \mathbb{R}_+$

$$(10) \quad D_{ab}(x, y) = \begin{cases} \left(\frac{\exp F(a, x, y)}{\exp F(b, x, y)} \right)^{\frac{1}{a-b}} = \exp \frac{F(a, x, y) - F(b, x, y)}{a-b} & \text{if } a - b \neq 0, \\ \exp f(a, x, y) & \text{if } a - b = 0. \end{cases}$$

Using the identity

$$\int_0^1 f(vb + (1-v)a, x, y) dv = \begin{cases} \frac{F(a, x, y) - F(b, x, y)}{a-b} & \text{if } a - b \neq 0, \\ f(a, x, y) & \text{if } a - b = 0, \end{cases}$$

we have

$$d(a, b, x, y) = D_{ab}(x, y) = \exp \int_0^1 f(vb + (1-v)a, x, y) dv \quad (a, b, x, y) \in R,$$

which shows that the function d is in $C^\infty(R)$ proving (iii). $\square$

Lemma 2. *Let $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be an arbitrary homogeneous function of degree one (i.e. $M(tx, ty) = tM(x, y)$ holds for all $x, y, t \in \mathbb{R}_+$). Then Minkowski's inequality*

$$(11) \quad M(x_1 + x_2, y_1 + y_2) \leq M(x_1, y_1) + M(x_2, y_2) \quad (x_1, x_2, y_1, y_2 \in \mathbb{R}_+)$$

(or its reverse inequality) holds if and only if the function m defined by $m(u) = M(u, 1)$ is a convex (or concave) function on $\mathbb{R}_+$.

If m is a strictly convex (or strictly concave) function on $\mathbb{R}_+$, then equality occurs in (11) (or in its reverse inequality) if and only if

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}.$$

Proof. Since $M(x, y) = yM\left(\frac{x}{y}, 1\right) = ym\left(\frac{x}{y}\right)$, we can write (11) as

$$m\left(\frac{y_1}{y_1+y_2} \frac{x_1}{y_1} + \frac{y_2}{y_1+y_2} \frac{x_2}{y_2}\right) \leq \frac{y_1}{y_1+y_2} m\left(\frac{x_1}{y_1}\right) + \frac{y_2}{y_1+y_2} m\left(\frac{x_2}{y_2}\right)$$

or, with the notations $\frac{x_1}{y_1} = u \in \mathbb{R}_+$, $\frac{x_2}{y_2} = v \in \mathbb{R}_+$, $\frac{y_1}{y_1+y_2} = \lambda \in (0, 1)$,

$$(12) \quad m(\lambda u + (1-\lambda)v) \leq \lambda m(u) + (1-\lambda)m(v),$$

which is exactly the convexity of m . If m is strictly convex, then equality stands in (12) or in (11) if and only if $u = v$ (see e.g. [9]). The proof for the reverse inequality is similar. $\square$

The next lemma is well-known again (see e.g. [9]).

Lemma 3. Let I be an open subinterval of $\mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be a function of class $C^2(I)$. The function g is convex [concave] if and only if $g''(x) \geq 0$ ($x \in I$) [$g''(x) \leq 0$ ($x \in I$)].

If $g''(x) > 0$ ($x \in I \setminus A$) [$g''(x) < 0$ ($x \in I \setminus A$)], where A is a finite subset of I , then g is strictly convex [concave], that is, in the convexity inequality

$$g(\lambda u + (1-\lambda)v) \leq \lambda g(u) + (1-\lambda)g(v)$$

[or in its reverse] equality holds if and only if $\lambda \in \{0, 1\}$, $u, v \in I$ or $\lambda \in [0, 1]$, $u = v \in I$.

Proof of Theorem 1. By Lemmas 1–3 (3) holds if and only if $m''_{a,b}(u) \geq 0$ ($u \in \mathbb{R}_+$), where $m_{a,b}$ is the continuous (moreover C^∞) extension of the function

$$D_{a,b}(u, 1) = \left(\frac{b(u^a - 1)}{a(u^b - 1)}\right)^{\frac{1}{a-b}} = \sqrt{u} \left(\frac{b \sinh(a \ln \sqrt{u})}{a \sinh(b \ln \sqrt{u})}\right)^{\frac{1}{a-b}}$$

$$(u \in \mathbb{R}_+, ab(a-b)(u-1) \neq 0)$$

to $a, b \in \mathbb{R}, u \in \mathbb{R}_+$. Using logarithmic differentiation, we obtain after some calculation that

$$(13) \quad \frac{4u^2 m''_{a,b}(u)}{m_{a,b}(u)} = \frac{a^2(1-a+b)}{(a-b)^2} (\coth^2 as - 1) + \frac{b^2(1+a-b)}{(a-b)^2} (\coth^2 bs - 1) - \frac{2ab}{(a-b)^2} (\coth as \coth bs - 1),$$

where $s = \ln \sqrt{u} \in \mathbb{R}$ and $ab(a-b)s \neq 0$. We need to write (13) in other forms, too.

The multiplication of (13) by

$$2 \frac{\sinh^2 as}{(as)^2} \frac{\sinh^2 bs}{(bs)^2}$$

removes the singularity $s = 0$ in (13). Applying the addition formula of cosh and the identity $\cosh^2 x - \sinh^2 x = 1$ we obtain

$$G_{a,b}(s) := \frac{8u^2 m''_{a,b}(u)}{m_{a,b}(u)} \frac{\sinh^2 as}{(as)^2} \frac{\sinh^2 bs}{(bs)^2} = \frac{2a^2(1-a+b) \sinh^2 bs + 2b^2(1+a-b) \sinh^2 as - 4ab \sinh as \sinh bs \cosh(a-b)s}{(ab(a-b)s)^2}.$$

Using the half-angle formula for $\sinh^2$, the “product into sum” formula for hyperbolic functions and the identities $a^2(1-a+b)-ab = a(1-a)(a-b)$, $b^2(1+a-b)-ab = -b(1-b)(a-b)$ we can write

$$(14) \quad G_{ab}(s) = \frac{(1-a)(\cosh 2bs - 1)}{ab^2(a-b)s^2} - \frac{(1-b)(\cosh 2as - 1)}{a^2b(a-b)s^2} + \frac{\cosh 2(a-b)s - 1}{ab(a-b)^2s^2}.$$

From (14) we easily obtain the Taylor series of G_{ab} :

$$(15) \quad G_{ab}(s) = \sum_{n=2}^{\infty} g_n(a, b) \frac{(2)^{2n} s^{2n-2}}{(2n)!} \quad (s \in \mathbb{R}),$$

where

$$(16) \quad g_n(a, b) = \frac{(1-a)b^{2n-1} - (1-b)a^{2n-1} + (a-b)^{2n-1}}{ab(a-b)} \quad (n = 2, 3, \dots).$$

One can easily see that g_n is a polynomial of a, b ; hence it can be defined for all $a, b \in \mathbb{R}$. Since the factors $\frac{\sinh^2 as}{(as)^2}$, $\frac{\sinh^2 bs}{(bs)^2}$ have the limit 1 at $a = 0, b = 0, s = 0$, the series development (15) of G_{ab} is valid for all $a, b, s \in \mathbb{R}$.

Now we are ready to prove that the conditions (4) are necessary for (3) to hold. By Lemma 3, (3) implies that $G_{ab}(s) \geq 0$. Hence

$$g_2(a, b) = \lim_{s \rightarrow 0} \frac{G_{ab}(s)}{s^2} \geq 0.$$

This gives the condition

$$(17) \quad g_2(a, b) = \frac{(1-a)b^3 - (1-b)a^3 + (a-b)^3}{ab(a-b)} = a + b - 3 \geq 0.$$

Multiplying G_{ab} by a suitable factor and then taking its limit at $s = \infty$, we can get another necessary condition for (3). From (14) one can see that $G_{ab}(s)s^2$ is a linear combination of four cosh terms (the fourth one being $\cosh 0s$). The next lemma will help to find the suitable factor for the limiting process.

Lemma 4. *Let*

$$\begin{aligned} H(s) &:= \sum_{k=1}^4 A_k(a_1, a_2, a_3, a_4) \cosh a_k s & (s \in \mathbb{R}), \\ c &:= \max\{|a_1|, |a_2|, |a_3|, |a_4|\}, \end{aligned}$$

where A_k are arbitrary functions of the real variables a_1, a_2, a_3, a_4 . Assume that $|a_1|, |a_2|, |a_3|, |a_4|$ are pairwise distinct numbers. Then

$$\lim_{s \rightarrow \infty} 2e^{-cs} H(s) = A_l(a_1, a_2, a_3, a_4) \quad \text{if } c = |a_l|.$$

Proof. Our statement easily follows from the limit relation

$$\lim_{s \rightarrow \infty} 2e^{-cs} \cosh \alpha s = \begin{cases} 1 & \text{if } c = |\alpha|, \\ 0 & \text{if } c > |\alpha|. \end{cases}$$

□

Let us return to the proof of necessity. By (i) of Lemma 1, we may assume, without restricting the generality, that $a \geq b$. This and (17) show that the point

(a, b) is either in the first or in the fourth quadrant. We claim that the latter case: $a > 0, b < 0$ is not possible. Assuming it, we would get from (13), that

$$\lim_{u \rightarrow \infty} \frac{4u^2 m''_{ab}(u)}{m_{ab}(u)} = \frac{4ab}{(a-b)^2} \geq 0$$

which is impossible. Therefore $a \geq 0, b \geq 0$.

Case 1: $a > b > 0$.

Since $\max\{2|a|, 2|b|, 2|a-b|, 0\} = 2|a| = 2a$, by Lemma 4, we get that

$$\lim_{s \rightarrow \infty} 2s^2 G_{ab}(s) e^{-2as} = -\frac{1-b}{a^2 b(a-b)} \geq 0$$

which implies that

$$(18) \quad b \geq 1.$$

The inequalities (17) and (18) give exactly the condition (4) in Case 1.

Case 2: $a > b = 0$.

From (14) we obtain for $a \neq 0$ that

$$G_{a0}(s) = \lim_{b \rightarrow 0} G_{ab}(s) = \frac{2a^2(1-a)s^2 + (a+1)(\cosh 2as - 1) - 2as \sinh 2as}{a^4 s^2}$$

and

$$\lim_{s \rightarrow \infty} 2s^2 G_{a0}(s) e^{-2as} = -\infty,$$

which shows that (3) cannot hold if $a > b = 0$.

Thus, if $a \geq b \geq 0$, then the necessary conditions for (3) are: (17) and (18). Extending the region given by (17), (18) symmetrically to the line $a = b$, we get exactly (4).

Next we prove that the conditions (4) are sufficient for (3) to hold. For symmetry reasons, we may assume again that $a \geq b \geq 0$. The region S determined by (4) and $a \geq b \geq 0$ can be written as $S_1 \cup S_2$, where

$$\begin{aligned} S_1 &= \{(a, b) \mid 3 \leq a+b, 1 \leq b < \frac{a}{2}\}, \\ S_2 &= \{(a, b) \mid 3 \leq a+b, \frac{a}{2} \leq b \leq a\}. \end{aligned}$$

We shall show by induction that, for $(a, b) \in S$,

$$g_n(a, b) \geq 0 \quad (n = 2, 3, \dots).$$

This implies that $G_{ab}(s) \geq 0$ if $(a, b) \in S$ and then, by Lemmas 1–3, (3) follows. We have

$$g_2(a, b) = a + b - 3 \geq 0 \quad \text{if } (a, b) \in S$$

and for all $(a, b) \in \mathbb{R}^2$

$$(19) \quad g_{n+1}(a, b) = (a-b)^2 g_n(a, b) + h_n(a, b) \quad (n = 1, 2, \dots),$$

where $g_1(a, b) = 0$ and

$$(20) \quad h_n(a, b) = \frac{a^{2n-2}(b-1)(2a-b) + b^{2n-2}(a-1)(a-2b)}{a-b} \quad (n = 1, 2, \dots).$$

It is clear that h_n is a polynomial in a and b .

If $(a, b) \in S_1$, then $a \geq 2, a-2b > 0, 2a-b = a + (a-b) \geq 0$. Therefore both terms of h_n are nonnegative and so is $h_n(a, b)$. By (19) this implies by induction that $g_n(a, b) \geq 0$.

If $(a, b) \in S_2$, then $2b - a \geq 0, a \geq b \geq 1, a + b \geq 3$ and in the identity

$$(21) \quad h_{n+1}(a, b) = a^2 h_n(a, b) + (a - 1)(2b - a)(a + b)b^{2n-2}$$

the second term is nonnegative since each of its factors are such. On the other hand, $h_1(a, b) = a + b - 1 \geq 0$; therefore, by induction $h_n(a, b) \geq 0$ and by (19), we get $g_n(a, b) \geq 0$.

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. The conditions (6) are necessary. We may suppose that $a \geq b$. We are going to prove that (4) implies $b \leq \min\{1, 3 - a\}$. By Lemma 2 and Lemma 3, $G_{ab}(s) \leq 0$ ($s \in \mathbb{R}$). Thus

$$(22) \quad \lim_{s \rightarrow 0} \frac{G_{ab}(s)}{s^2} = g_2(a, b) = a + b - 3 \leq 0.$$

If $a \geq b > 0$ we can derive another necessary condition.

Case 1: $a > b > 0$.

Similarly to the proof of Theorem 1, we get, by Lemma 4, that

$$\lim_{s \rightarrow \infty} 2s^2 G_{ab}(s) e^{-2as} = -\frac{1 - b}{a^2 b(a - b)} \leq 0;$$

therefore,

$$(23) \quad b \leq 1$$

Case 2: $a = b > 0$.

From (14), we get (for $a \neq 0$) that

$$G_{aa}(s) = \lim_{b \rightarrow a} G_{ab}(s) = \frac{2a^2 s^2 - (2a - 1)(\cosh 2as - 1) - 2a(1 - a)s \sinh 2as}{a^4 s^2};$$

hence

$$\lim_{s \rightarrow \infty} 2s G_{aa}(s) e^{-2as} = -2a(1 - a) \leq 0.$$

This gives the same necessary condition $b = a \leq 1$, completing the proof of necessity.

To prove the sufficiency of (6), we assume again that $a \geq b$ and decompose the region T determined by (6) and $a \geq b$ into the union of the following six sets:

$$\begin{aligned} T_1 &= \{(a, b) \mid a \leq 0, b \leq 0, a \geq b\}, \\ T_2 &= \{(a, b) \mid a \geq 1, b \leq \frac{a}{2}, 0 \leq a + b \leq 3\}, \\ T_3 &= \{(a, b) \mid 0 < a < 1, -a \leq b \leq \frac{a}{2}\}, \\ T_4 &= \{(a, b) \mid a \geq 1, \frac{a}{2} < b \leq 1\}, \\ T_5 &= \{(a, b) \mid 0 < a < 1, \frac{a}{2} < b \leq a\}, \\ T_6 &= \{(a, b) \mid a > 0, b < -a\}. \end{aligned}$$

We shall prove that for $(a, b) \in T$ the coefficients $g_n(a, b)$ are nonpositive. Hence $G_{ab}(s) \leq 0$, which implies (4). Let $h_n(a, b)$ be defined by (20). Then we have (19).

If $(a, b) \in T_1 \cup T_2$, then

$$h_1(a, b) = a + b - 3 \leq 0$$

and in the identity

$$(24) \quad h_{n+1}(a, b) = a^2 h_n(a, b) + (a - 1)(2b - a)(a + b)b^{2n-2}$$

the second term of the right-hand side is nonpositive. In T_1 , we have $a - 1 < 0, a + b \leq 0, b \leq a \leq \frac{a}{2}$, while in T_2 the inequalities $a - 1 \geq 0, a + b \geq 0, b \leq \frac{a}{2}$

hold. Hence, by induction, it follows that $h_n(a, b) \leq 0$ ($n = 1, 2, \dots$) and by (19), we get, again by induction, the required inequality

$$(25) \quad g_n(a, b) \leq 0 \quad (n = 2, 3, \dots).$$

If $(a, b) \in T_3 \cup T_4$ then both terms of

$$(26) \quad h_n(a, b) = \frac{a^{2n-2}(b-1)(2a-b)}{a-b} + \frac{b^{2n-2}(a-1)(a-2b)}{a-b} \quad (n = 1, 2, \dots)$$

are nonpositive. In $T_3 \cup T_4$ we have $b-1 \leq 0$, $2a-b \geq 0$, and thus the first term of h_n is nonpositive. In T_3 we have $a-1 \leq 0$, $a-2b \geq 0$ while in T_4 the inequalities $a-1 \geq 0$, $a-2b < 0$ hold; hence in both cases the second term of h_n is nonpositive. Using (19) we get, by induction, the inequality (25).

If $(a, b) \in T_5$, then in the identity

$$g_{n+1}(a, b) = a^2 g_n(a, b) + \frac{a+b}{a} [(a-1)b^{2n-2} + (b-2a)(a-b)^{2n-3}] \quad (n = 1, 2, \dots)$$

both terms in the bracket are nonpositive, the factor in front of the bracket is positive, and hence, by induction, we get (25).

If $(a, b) \in T_6$, then $-b \geq a > 0$,

$$(a-b)^{2n-1} \geq a^{2n-1} + (-b)^{2n-1}.$$

Hence

$$\begin{aligned} (1-a)b^{2n-1} - (1-b)a^{2n-1} + (a-b)^{2n-1} &\geq \\ (a-1)(-b)^{2n-1} - (1-b)a^{2n-1} + a^{2n-1} + (-b)^{2n-1} &= \\ a(-b)^{2n-1} + ba^{2n-1} = ab[a^{2n-2} - b^{2n-2}] &\geq 0 \end{aligned}$$

and by $ab \leq 0$

$$g_n(a, b) = \frac{(1-a)b^{2n-1} - (1-b)a^{2n-1} + (a-b)^{2n-1}}{ab(a-b)} \leq 0 \quad (n = 2, 3, \dots),$$

proving (25).

Thus, we have shown that (25) holds on the whole set T which proves the sufficiency of conditions (6). $\square$

Proof of Theorem 3. Suppose that (3) holds. Then (4) is also true by Theorem 1.

If $b > 3-a$ in (4), then $g_2(a, b) = a+b-3 > 0$, the other coefficients g_n ($n = 3, 4, \dots$) are nonnegative, and hence $G_{ab}(s) > 0$ for $s \neq 0$ which implies the strict convexity of m_{ab} on $\mathbb{R}_+$. By Lemma 2 equality holds in (3) if and only if (8) is satisfied.

If $b = 3-a$ in (4), then $g_2(a, b) = a+b-3 = 0$. By assumption $a \neq 1$, $a \neq 2$; thus, we have $1 < a < 2$, $b = 3-a$. We claim that for these pairs (a, b) the coefficient g_3 is positive. A simple calculation shows that

$$g_3(a, b) = (a+b)^3 - 5(a+b)^2 - 2ab(a+b) + 15ab,$$

and therefore,

$$g_3(a, 3-a) = -9(a-1)(a-2) > 0 \text{ if } 1 < a < 2.$$

The other coefficients $g_n(a, b)$ ($n = 4, 5, \dots$) are nonnegative in the region (4); hence $G_{ab}(s) > 0$ for $s \neq 0$, which implies the strict convexity of m_{ab} on $\mathbb{R}_+$. By Lemma 2, equality occurs in (3) if and only if (8) holds.

If (5) holds, then by Theorem 2 (6) is true. If $a + b < 3$ in (6), then $g_2(a, b) < 0$, $g_n(a, b) \leq 0$ ($n = 3, 4, \dots$). If $a + b = 3$ in (6), then either $a < 1$, or $a > 2$, in both cases $g_3(a, 3 - a) = -9(a - 1)(a - 2) < 0$, and $g_n(a, 3 - a) \leq 0$ ($n = 4, 5, \dots$). Therefore $G_{ab}(s) > 0$, m_{ab} is strictly concave and, by Lemma 2, equality occurs in (5) if and only if (8) is valid. $\square$

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