METRIZABILITY OF SEQUENTIAL TOPOLOGICAL GROUPS WITH POINT-COUNTABLE k -NETWORKS

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(Communicated by Franklin D. Tall)

ABSTRACT. We prove that a Hausdorff sequential topological group with a point-countable k -network is metrizable iff its sequential order is less than ω_1 . In the non Hausdorff case metrizability may be replaced by σ -locally finite base.

1. INTRODUCTION

A family γ of subsets of a topological space X is said to be a k -network if for any compact $K \subseteq X$ and any open $U \supseteq K$ there is a finite $\gamma_K \subseteq \gamma$ such that $K \subseteq \cup \gamma_K \subseteq U$. This notion has found several applications in the generalized metric spaces theory (see [GMT] and references there). Usually the notion of k -network is studied in combination with another property, namely point-countability. Recall that a family γ is called *point-countable* if any point of X belongs to at most countably many elements of γ .

In the present paper we study sequential topological groups with point-countable k -networks. We obtain a metrization criterion for such groups in terms of sequential order. P. Nyikos in [N] asked if the sequential order of a sequential topological group is equal to ω_1 if the group is not Fréchet. For general topological groups the answer is consistently negative [S]. We show that in the case of groups with point-countable k -networks it is possible even to strengthen P. Nyikos' hypothesis. Namely we prove that a Hausdorff sequential topological group with a point-countable k -network is metrizable if its sequential order is not equal to ω_1 . In [A] A. Arhangel'skii proved that a Fréchet topological group is metrizable provided it has a countable k -network.

Some useful constructions give topological groups which have point-countable k -networks. Thus it follows from [A] that the free topological group over a metrizable compact has a countable k -network. In paper [ZP] on every infinite Abelian group a sequential non Fréchet topology is introduced. It may be shown that the resulting topological group has a point-countable k -network. So it follows from our criterion that every infinite Abelian group admits a sequential topology with the sequential order ω_1 . We also consider the general non Hausdorff case where metrizability is replaced by σ -locally finite base. Non Hausdorff groups arise naturally as quotients via nonclosed subspaces.

Received by the editors May 10, 1995 and, in revised form, September 15, 1996.

1991 *Mathematics Subject Classification*. Primary 54H11; Secondary 54D55, 54A20, 54E35.

Key words and phrases. Topological group, sequential space, sequential order, Fréchet space, k -network.

Let us now give some definitions. If A is a subset of a space X , then $[A]^{seq}$ denotes the *sequential closure* of A , i.e. the set of limits of convergent sequences in A . Obviously $A \subseteq [A]^{seq}$. We define $[A]_\alpha$ by induction on $\alpha \in \omega_1 + 1$ as follows: $[A]_0 = A$, $[A]_{\alpha+1} = [[A]_\alpha]^{seq}$ and $[A]_\alpha = \bigcup \{ [A]_\beta \mid \beta < \alpha \}$ for a limit α . One can easily see that $[A]_{\omega_1+1} = [A]_{\omega_1}$, and that a space X is sequential if and only if $\overline{A} = [A]_{\omega_1}$ for every $A \subset X$. For a sequential space X we define $so(X)$, the *sequential order* of X , by $so(X) = \min \{ \alpha \in \omega_1 + 1 \mid \overline{A} = [A]_\alpha \text{ for every } A \subset X \}$. Note that a space is Fréchet if and only if it is a sequential space with sequential order 1. Consider the set $S = \omega^2 \cup \omega \cup \{\omega\}$. Define a topology on S as follows. Every point of ω^2 is isolated, a typical neighborhood of n is $\{n\} \cup (\{n\} \times \omega \setminus \text{finitely many points})$, $U \ni \{\omega\}$ is open if $(U \cap \{n\} \times \omega) \cup \{n\}$ is a neighborhood of n for every $n \in U$ and $\omega \setminus U$ is finite. The resulting space is called *Arens' space* S_2 . Identifying the limit points of the sum of countably many converging sequences and equipping the resulting space with the quotient topology we get another canonical space called *sequential fan* S_ω . A subset $\sigma \subseteq \omega^2$ is called *thin* if $|\sigma \cap \{n\} \times \omega| < \aleph_0$ for any $n \in \omega$. Define $\omega(n) = \{i \mid i > n\}$.

2. SEQUENTIAL TOPOLOGICAL GROUPS WITH POINT-COUNTABLE k -NETWORKS

We shall prove some technical lemmas first.

Lemma 2.1. *A Hausdorff sequential non Fréchet space with a point-countable k -network contains a closed copy of S_2 .*

Proof. Let X be a sequential non Fréchet space with a point-countable k -network γ . Then there exists an injection $t : \omega^2 \rightarrow X$ such that $t(n, k) \rightarrow x_n$ as $k \rightarrow \infty$, $x_i \neq x_j$ if $i \neq j$, $x_i \rightarrow x \in X$ as $i \rightarrow \infty$ and there is no sequence in $t(\omega^2)$ converging to x . Put $\{\xi_i \mid i \in \omega\} = \{\xi \in \gamma \mid \xi \cap t(\omega^2) \neq \emptyset, x \notin \overline{\xi}\}$. Then by induction choose $n_i \in \omega$ and $k_i \in \omega$ for every $i \in \omega$ so that $\{t(n_i, j) \mid j \geq k_i\} \subseteq X \setminus \bigcup_{j \leq i} \xi_j$. It is now easy to see that any set of the form $\{t(n_i, p_i) \mid i \in \omega\}$ where $p_i \geq k_i$ is a closed discrete subset of X . Indeed, were it not both it would have a convergent subsequence, so assume without loss of generality that $\langle t(n_i, p_i) : i \in \omega \rangle$ converges to p . Since $p \neq x$, there exists ξ_j containing infinitely many $t(n_i, p_i)$. But this contradicts the way the points $t(n_i, p_i)$ were chosen. It follows that $\{t(n_i, j) \mid i \in \omega, j \geq k_i\} \cup \{x_{n_i} \mid i \in \omega\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2 . $\square$

Let us introduce a property stronger than the one described in Lemma 2.1:

$P(x, U)$ If $U \subseteq X$, $\{x_i\}_{i \in \omega} \subseteq U$, $x_i \rightarrow x$ as $i \rightarrow \infty$ and $x_i \neq x_j$ if $i \neq j$ then there is $\Gamma = \{x(n, k) \mid n, k \in \omega\} \subseteq U$ such that $x(n, k) \rightarrow x_n$ as $k \rightarrow \infty$, $t : \omega^2 \rightarrow \Gamma$ is a bijection where $t(n, k) = x(n, k)$ and $\Gamma \cup \{x_i \mid i \in \omega\} \cup \{x\}$ is a closed subset of G homeomorphic to S_2 .

Lemma 2.2. *Let G be a Hausdorff non Fréchet sequential topological group with a point-countable k -network. Then for any $x \in G$ and any open $U \subseteq G$ the property $P(x, U)$ takes place.*

Proof. By non Fréchetness of G and Lemma 2.1 there is a closed subset of G homeomorphic to S_2 . So by [NST2, Lemma 2.1] G contains a closed subset homeomorphic to S_ω . Let $y(n, k) \rightarrow e$ as $k \rightarrow \infty$ where e is the unit of G and $y(n, k) \neq y(l, m)$ if $(n, k) \neq (l, m)$ and the set $\{y(n, k) \mid n, k \in \omega\} \cup \{e\}$ be a closed subset of G naturally homeomorphic to S_ω . Let U be open in G , $\{x_i\}_{i \in \omega} \subseteq U$, $x_i \rightarrow x$ as $i \rightarrow \infty$ and

$x_i \neq x_j$ if $i \neq j$. Denote $x(i, k) = x_i \cdot y(i, k)$ for $i, k \in \omega$. Then $x(i, k) \rightarrow x_i$ as $k \rightarrow \infty$ so it is possible to choose $k_i \in \omega$ for every $i \in \omega$ so that $\{x(i, k) \mid i \in \omega, k \geq k_i\} \subseteq U$ and $x(i, k) \neq x(i', k')$ if $(i, k) \neq (i', k')$ and $k \geq k_i, k' \geq k_{i'}$. To prove $P(x, U)$ it is enough to show that $\{x_i \mid i \in \omega\} \cup \{x(i, k) \mid i \in \omega, k \geq k_i\} \cup \{x\}$ is a closed in G subset homeomorphic to S_2 . Suppose the contrary. Then there is a sequence $\langle x(i_j, k_j) : j \in \omega \rangle$ converging to $p \in G$ such that $i_j \neq i_{j'}$ if $j \neq j'$. Now $y(i_j, k_j) = x_{i_j}^{-1} \cdot x(i_j, k_j)$ so $y(i_j, k_j) \rightarrow x^{-1} \cdot p$ as $j \rightarrow \infty$ contradicting the way the points $y(n, k)$ were chosen. $\square$

Lemma 2.3. *Let X be a regular sequential space with a point-countable k -network such that for any $x \in X$ and $U \subseteq X$ the property $P(x, U)$ holds. Then for any $\alpha < \omega_1$, $x \in X$, $U \subseteq X$ open in X the following property holds:*

$Q(\alpha, x, U)$: *If $\{x_i\}_{i \in \omega} \subseteq U$, $x_i \rightarrow x$ as $i \rightarrow \infty$ then there is $Q \subseteq U$ such that $\overline{Q}$ is countable, $\overline{Q} \setminus \{x\} \subseteq U$, $x \in [Q]_\alpha$, $x \notin [Q]_\beta$ for $\beta < \alpha$.*

Proof. Suppose we have already proved $Q(\beta, x, U)$ for any $x \in X$, $U \subseteq X$ open in X and $\beta < \alpha$. First suppose that $\alpha = \beta + 1$ for some $\beta < \omega_1$. Let us note the following fact:

Fact. *Let $t : \omega^2 \rightarrow \Gamma = \{x(n, k) \mid x(n, k) = t((n, k))\}$ be a bijection. Let $\delta = \{\xi_n \mid n \in \omega\}$ be a family of subsets of X . Then there exists a thin set $\sigma(\delta) \subseteq \omega^2$ such that for any $\xi_k \in \delta$ either $\xi_k \cap t(\sigma(\delta)) \neq \emptyset$ or $\xi_k \cap t(\omega(m) \times \omega) = \emptyset$ for some $m \in \omega$.*

Let $\{x_i\}_{i \in \omega} \subseteq U$. Using $P(x, U)$ choose $t : \omega^2 \rightarrow \Gamma \subseteq U \setminus \{x\}$ a bijection so that $t(n, k) \rightarrow x_n$ as $k \rightarrow \infty$, $x_n \neq x_k \neq x$ if $n \neq k$ and $\Gamma \cup \{x_n \mid n \in \omega\} \cup \{x\}$ is a closed subset of X homeomorphic to S_2 . Let $S \subseteq X$ be countable. Let us denote

$$\gamma(S) = \{\xi \mid \xi \in \gamma, \xi \cap S \neq \emptyset, \overline{\xi} \cap t(\omega(m) \times \omega) = \emptyset \text{ for some } m \in \omega\}.$$

Now $\gamma(S)$ is countable so put $\gamma(S) = \{\xi_n(S) \mid i \in \omega\}$. Let us now construct the sets $Q_n \subseteq U$ such that:

(1) $\overline{Q_i}$ is countable, $\overline{Q_i} \cap (\{x_j \mid j \neq n_i\} \cup \{x\}) = \emptyset$, $x_{n_i} \in [Q_i]_\beta$, $x_{n_i} \notin [Q_i]_\gamma$ if $\gamma < \beta$, $n_{i+1} > n_i$,

(2) $\overline{Q_i} \setminus \{x_{n_i}\} \subseteq (X \cap U) \setminus \bigcup_{n, k < i} \overline{\xi_n(Q_k)}$.

Let us first show how to arrange for this. Suppose we have already chosen $n_i \in \omega$, $Q_i \subseteq X$ so that they satisfy (1)–(2) for all $i < k$. Consider the family

$$\Delta = \{\overline{\xi_i(Q_j)} \mid i, j < k\}.$$

Δ is finite. By the construction of $\xi_i(\overline{Q_j})$ for every $i < k$, $j < k$ there is $m(i, j)$ such that

$$\overline{\xi_i(Q_j)} \cap t(\omega(m(i, j)) \times \omega) = \emptyset.$$

Put $m = \max(\{m(i, j) \mid i < k, j < k\} \cup \{n_j \mid j < k\})$. Then it is easy to see that $(\bigcup \Delta) \cap t(\omega(m) \times \omega) = \emptyset$. So we have that $x(m+1, i) \rightarrow x_{m+1}$ as $i \rightarrow \infty$, $\{x(m+1, i) \mid i \in \omega\} \subseteq (X \cap U) \setminus \bigcup \Delta$. Thus by $Q(\beta, x_{m+1}, ((X \cap U) \setminus \bigcup \Delta))$ there is Q_k such that if $x_{n_k} = x_{m+1}$ then (1) takes place and $\overline{Q_k} \setminus \{x_{n_k}\} \subseteq (X \cap U) \setminus \bigcup \Delta$ that is (2) takes place.

Now consider the set $Q' = \bigcup_{i \in \omega} \overline{Q_i} \cup \{x\}$. Using the fact find the set $D = t(\sigma(\gamma(Q')))$.

Claim. *If $z_i \in \overline{Q_i} \setminus \{x_{n_i}\}$ and $z_{k_i} \rightarrow z$ as $i \rightarrow \infty$ then $z \in D$; in particular $Q' \cup D$ is closed in X .*

Proof of claim. Suppose the contrary; then $z \in V = X \setminus D$ and V is open in X because D is a closed discrete subset of X by the construction of t and Γ . Using the regularity of X choose $\xi \in \gamma$ such that $\bar{\xi} \subseteq V$ and $\xi \cap \{z_{n_i} \mid i \in \omega\}$ is infinite. Then $\bar{\xi} \cap t(\sigma(\gamma(Q')))) = \emptyset$ and thus $\bar{\xi} \cap t(\omega(m) \times \omega) = \emptyset$ for some $m \in \omega$ by the choice of $\sigma(\gamma(Q'))$. So $\xi = \xi_n(\bar{Q}_k)$ for some $n \in \omega$, $k \in \omega$. But then $z_{k_i} \notin \xi$ if $k_i > \max\{n+1, k+1\}$ by (2). A contradiction. $\square$

Consider the set $Q = \bigcup_{i \in \omega} Q_i$. It follows from the claim that $\bar{Q} \setminus \{x\} \subseteq U$. Also $x \in [Q]_{\beta+1} = [Q]_\alpha$. Let $z_i \rightarrow x$ as $i \rightarrow \infty$ and $z_i \in [Q]_{\gamma_i}$ for some $\gamma_i \leq \beta$. Using the claim and the fact that D is a closed discrete subset of X we may assume that $z_i \in [Q_{k_i}]_{\gamma_i}$ and $k_{i+1} > k_i$. Since $x \notin D \subseteq \Gamma$ it follows from the claim that for some $k \in \omega$ $\{z_i \mid i \geq k\} \subseteq \{x_{n_i} \mid i \in \omega\}$. It implies that $\gamma_i \geq \beta$ by (1). So $Q(\alpha, x, U)$ holds. The case of a limit α may be considered similarly. $\square$

Theorem 2.4. *Let G be a Hausdorff sequential topological group with a point-countable k -network. If $so(G) < \omega_1$ then G is metrizable.*

Proof. Suppose $so(G) = \alpha$. Let us first show that G is Fréchet. Suppose not. Then by Lemmas 2.2–2.3 property $Q(\alpha+1, e, G)$ takes place. Obviously $Q(\alpha+1, e, G)$ implies $so(G) \geq \alpha+1 > \alpha$. A contradiction. Now G is a Fréchet topological group with a point-countable k -network. By [N] it is an α_4 -space and thus is a countably bi- k -space in the sense of [M2]. Now by [GMT, Corollary 3.6] G is first countable. Thus by the classical Birkhoff-Kakutani theorem G is metrizable. $\square$

Now let us consider the general case.

Theorem 2.5. *Let G be a (possibly non Hausdorff) sequential topological group with a point-countable k -network. If $so(G) < \omega_1$ then G has a σ -locally finite base.*

Proof. Let $E = \bar{e} \subseteq G$ be the closure of the unit in G . Then E is a closed normal subgroup of G . Let $f : G \rightarrow G/E$ be the natural quotient map. Then G/E is a sequential Hausdorff topological group. The map f has the property: for any $x \in G/E$ the subspace $f^{-1}(x)$ has antidiscrete topology. Let $g_x \in G$ be chosen so that $f(g_x) = x$. Then the property of f mentioned above gives that for any $\{x_i\}_{i \in \omega} \subseteq G/E$ such that $x_n \rightarrow x \in G/E$ as $n \rightarrow \infty$ holds $g_{x_n} \rightarrow g_x$ as $n \rightarrow \infty$. This yields that the sequential order of G/E is less than or equal to the sequential order of G . Put $G' = \{g_x \mid x \in G/E\}$. If γ is a point-countable k -network for G then the family $\{f(\xi \cap G') \mid \xi \in \gamma\}$ is point-countable. Using the property of f again one has that for any convergent sequence $S \subseteq G/E$ and its neighborhood U in G/E there is $\xi \in \gamma$ such that $f(\xi \cap G') \subseteq U$ and the set $f(\xi \cap G') \cap S$ is infinite. This gives that the family is a k -network.

If G has sequential order less than ω_1 then by Theorem 2.4 G/E is metrizable and thus has a σ -locally-finite base $\mathcal{B}$ by the Bing-Nagata-Smirnoff theorem. Now since the fibres of f are antidiscrete, then the preimage of $\mathcal{B}$ under f is a σ -locally-finite base for G . $\square$

The author wishes to thank Professor T. Nogura for advice and moral support and the referee for the suggestion to include the non Hausdorff case and many other useful remarks.

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