INCOHERENT NEGATIVELY CURVED GROUPS

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ABSTRACT. In part 1, a construction of Rips is modified so that it produces a $\text{CAT}(-1)$ group instead of a small-cancellation group. Thus, many of the applications of Rips' construction to small-cancellation groups may be applied to $\text{CAT}(-1)$ groups as well. Part 2 offers a simple way of producing incoherent groups.

INTRODUCTION

Given a finitely presented group Q , a construction of Rips' [10] provides a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

where G is a finitely presented small-cancellation group and N is a finitely generated normal subgroup, generated by two elements of G .

Applications of Rips' construction generally involve lifting some property (usually a 'subgroup pathology') from an arbitrary (usually finitely presented) group Q to the small-cancellation group G . For instance, in [10] Rips uses his construction to show that there are examples of small-cancellation groups that do not have the finitely generated intersection property. In [3], Rips' construction is used to show that various problems about small-cancellation groups are recursively unsolvable. In [2], Rips' construction is used to find Automatic groups with non-Automatic finitely-presented subgroups.

A group is said to be *coherent* [12] if every finitely generated subgroup is finitely presented. The question of whether or not $C(8)$ small-cancellation groups are coherent was resolved negatively by Rips using his construction [10]. The main objective of this paper is to answer a question of Gersten's [7], who asked if fundamental groups of negatively curved 2-complexes are coherent. Parts 1 and 2 of this paper provide methods for constructing incoherent negatively curved groups.

In part 1 of this paper, I provide a construction similar to that of Rips, but in which G is the fundamental group of a compact negatively curved 2-complex instead of a small-cancellation group.

1.3 Theorem: Adapted Rips construction. *Let Q be a finitely presented group. Then there exists a group G which is the fundamental group of a compact negatively curved 2-complex and a finitely generated normal subgroup $N \triangleleft G$ such that $G/N \cong Q$.*

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It should be mentioned that the 2-complexes corresponding to Rips' construction do not admit metrics of negative curvature. This construction also differs from that of Rips' in the number of generators of N . While Rips' normal subgroup N is generated by only 2 elements of G , the normal subgroup N of this paper is generated by many elements of G , and the number of such generators grows with the length of the presentation for Q . Allowing N to be generated by many elements instead of just 2, makes it easier to accomodate the somewhat stricter negative curvature condition.

Thus, the applications of Rips' construction in [2], [3], and [10] mentioned above may be extended to finding examples of fundamental groups of compact negatively curved 2-complexes (or where appropriate, non-positively curved complexes) exhibiting the various pathologies. For example, Corollary 1.5 provides certain pathological examples of negatively curved groups analogous to the pathological examples of small-cancellation groups given in [10]. In particular, it is shown how to construct an incoherent negatively curved group.

In part 2 of this paper, I provide a different method, which is more geometric, for constructing incoherent finitely presented groups (see Theorem 2.1). The essential idea is to build a group in such a way that it contains a finitely generated subgroup with (an easily determined) infinitely generated H_2 . This method is somewhat more transparent than that of using Rips' construction and I hope that it sheds some more light on incoherent groups.

Using this method, I provide an easy example of a compact negatively curved 2-complex with incoherent fundamental group. Note, that while the method of part 1 yields a plethora of such examples, they are all quite complicated.

It is worth mentioning that it is easy to give examples of compact non-positively curved 2-complexes with incoherent fundamental group. Indeed, $F_2 \times F_2$, the product of two free groups of rank 2, is incoherent, and is the fundamental group of a non-positively curved squared complex. See also [4].

The class of coherent groups is not very well understood. It includes certain small groups, like finite groups and nilpotent groups. Scott [11] proved that it includes all 3-manifold groups. Serre asks whether $SL_3(\mathbb{Z})$ is coherent. It is also asked [1] whether 1-relator groups are coherent. Note that Example 2.3 is a 2-relator group which is incoherent.

PART 1: ADAPTED RIPS' CONSTRUCTION

1.1 Definition: No 2-letter repetitions. A set of words $V_1, \dots, V_v$ in some alphabet $B = \{b_1, \dots, b_n\}$ is said to have no 2-letter repetitions if each 2-letter word in B appears as a subword of at most one of the V_i , and in at most one place in each of the V_i . (In the language of small-cancellation theory, this is nearly equivalent to no pieces of length ≥ 2 .)

1.2 Lemma: Long word with no 2-letter repetitions. *Let J be a positive integer, and let*

$$X_J = \{x_1, x_2, \dots, x_J\}$$

be a set of J letters. Then there is a (positive) word Σ_J in X_J of length J^2 which has the property that no 2-letter word in X_J appears more than once in Σ_J as a subword.

Proof. Consider the sequence

$$\Sigma_1 = (x_1)$$

$$\Sigma_2 = (x_1x_1x_2)(x_2)$$

$$\Sigma_3 = (x_1x_1x_2x_1x_3)(x_2x_2x_3)(x_3)$$

$$\Sigma_4 = (x_1x_1x_2x_1x_3x_1x_4)(x_2x_2x_3x_2x_4)(x_3x_3x_4)(x_4).$$

In general, define Σ_J to be the following word:

$$\Sigma_J = (x_1x_1x_2x_1x_3x_1x_4 \cdots x_1x_J)(x_2x_2x_3x_2x_4 \cdots x_2x_J) \cdots (x_{J-1}x_{J-1}x_J)(x_J).$$

It is easy to see that

$$\|\Sigma_J\| = \sum_{i=1}^J (2i-1) = J^2.$$

Furthermore, it is easy to see that Σ_J has no repetitions of a 2-letter subword. $\square$

1.3 Theorem: Adapted Rips construction. *Let Q be a finitely presented group. Then there exists a group G which is the fundamental group of a compact negatively curved 2-complex and a finitely generated normal subgroup $N \triangleleft G$ such that $G/N \cong Q$.*

Proof. Let Q be presented by

$$\langle q_1, \dots, q_I \mid R_1, \dots, R_K \rangle.$$

We will assume that each of the relators is a reduced word.

Following Rips' construction, we wish to choose a presentation for G of the form

$$\left\langle q_1, \dots, q_I \mid \begin{array}{l} x_j^{q_i} = W_{ij+}, \\ x_j^{q_i^{-1}} = W_{ij-}, \end{array} R_k = W_k \right\rangle \quad \left(\begin{array}{l} i \in \{1, \dots, I\} \\ j \in \{1, \dots, J\} \\ k \in \{1, \dots, K\} \end{array} \right)$$

where $W_{ij\pm}$ and W_k are positive words of length 14 and $2\|R_k\| + 8$, respectively, in the X_J letters, as will be described below. Also, we will postpone our choice of J and thus the number of additional generators, until after it is motivated below. Note that we use the notation $b^a = aba^{-1}$ to denote conjugation by a . One sees immediately that the finitely generated subgroup $N = \langle X_J \rangle$ is a normal subgroup of G , and furthermore $G/N \cong Q$.

We wish to show that with the appropriate choices, the standard 2-complex of the above presentation for G will have a metric of negative curvature. In order to motivate the choices of $W_{ij\pm}$ and W_k we consider the diagrams of Figure 1. We have divided the polygons in Figure 1, corresponding to the relations of the presentation of G , into pentagons.

If we were to metrize each of these pentagons as a regular hyperbolic pentagon with angle $\frac{\pi}{2}$, then any corner of our polygons containing a q -edge would have angle π . This greatly simplifies what must be verified in order for the link condition (see [6], [8]) to be satisfied. Since we have assumed above that the words R_i are reduced, there is no corner with label $q_i q_i^{-1}$. Therefore any cycle of corners containing some black (i.e., q -labelled) edge must have total angle sum $\geq 2\pi$. Thus it is sufficient to consider sequences of corners not containing any black edges.

FIGURE 1. The polygon on the left corresponds to a relation of type $x_j^{q_i} = W_{ij+}$. It is formed by gluing together 5 regular hyperbolic pentagons with angle $\frac{\pi}{2}$. The polygon on the right corresponds to a relation of type $R_k = W_k$. It is formed by gluing together $\|R_k\| + 2$ regular hyperbolic pentagons with angle $\frac{\pi}{2}$. In both figures, the black arrows label q edges, and the white arrows label x edges. The long words $W_{ij\pm}$ and W_k correspond to the long sequence of white edges.

Because we chose the words $W_{ij\pm}$ and W_k to be positive, the all-white (i.e., containing only x -labelled edges) corners have labels which are of the form $x_j x_h$ (or of the form $x_h^{-1} x_j^{-1}$). It is easy to see that any all-white cycle must have even length. However, since any corner has angle $\geq \frac{\pi}{2}$, it is sufficient to demand that there be no such cycles of length 2, that is, no two corners have the same $x_j x_h$ label. In other words, we need only choose the set of words $\{W_{ij+}, W_{ij-}, W_k \mid 1 \leq i \leq I, 1 \leq j \leq J, 1 \leq k \leq K\}$ so that it has no 2-letter repetitions.

Finally, we choose J to be an integer large enough so that J^2 exceeds the total length of the $2IJ$ distinct $W_{ij\pm}$ words of length 14, and the K distinct W_k words each of length $2\|R_k\| + 8$. So

$$(1) \quad J^2 \geq (2IJ)14 + \sum_{k=1}^K (2\|R_k\| + 8).$$

It now follows from the lemma that there is a (positive) word Σ_J in $\{x_1, \dots, x_J\}$ whose length is J^2 and which has no 2-letter repetitions. Because of equation (1), we see that Σ_J is long enough so that Σ_J can be chopped up into a set of words $\{W_1, \dots, W_k, W_{11+}, W_{11-}, \dots, W_{IJ+}, W_{IJ-}\}$ which again has no 2-letter repetitions. $\square$

1.4 Remark. One is tempted to subdivide the polygons of the complex constructed above into pentagons as suggested by the diagrams. The derived complex obviously satisfies the small cancellation conditions $C(5) - T(4)$. In fact, it is not difficult to show, that for any large n and m , one can produce a group G , designed to fit into the exact sequence above, so that G is π_1 of a negatively curved 2-complex formed from regular hyperbolic n -gons with angle $\frac{2\pi}{m}$. To do this, one chooses Σ_J to satisfy the $T(m)$ condition, and one subdivides the polygons of Figure 1 into n -gons in a slightly more complicated fashion - making sure that there are at least m n -gons meeting at each vertex incident with a black edge.

Following Rips, we have the following corollary of Theorem 1.3:

1.5 Corollary. *For suitable Q , we obtain a group G which is the fundamental group of a negatively curved 2-complex such that:*

- (1) *There are finitely generated subgroups H_1 and H_2 of G such that $H_1 \cap H_2$ is not finitely generated.*
- (2) *G is incoherent.*
- (3) *The generalized word problem in G is not solvable.*

Proof. We repeat the proofs given in [10]. Let ϕ denote the quotient map $G \rightarrow Q$ of Theorem 1.3. Note that if $K = \langle k_1, \dots, k_s \rangle$ is a finitely generated subgroup of Q , then its preimage in G , $H = \phi^{-1}(K)$, is also finitely generated. Indeed, $H = \langle h_1, \dots, h_s, x_1, \dots, x_j \rangle$ where for each i , $\phi(h_i) = k_i$.

- (1) Let Q be a finitely presented group with a pair of finitely generated subgroups K_1 and K_2 whose intersection $K_1 \cap K_2$ is not finitely generated. Let $H_1 = \phi^{-1}(K_1)$ and $H_2 = \phi^{-1}(K_2)$. Then H_1 and H_2 are finitely generated subgroups of G , but $H_1 \cap H_2$ is infinitely generated because $\phi(H_1 \cap H_2) = K_1 \cap K_2$.
- (2) Let Q be a finitely presented incoherent group; then G is incoherent. This is because if $K \subset Q$ is a finitely generated subgroup which is not finitely presented, then $H = \phi^{-1}(K)$ is finitely generated but not finitely presented because $H/N \cong K$.
- (3) Let Q be a finitely presented group with unsolvable word problem; then the generalized word problem for G with respect to N is not solvable. This is because an algorithm which decides whether or not an element of G is an element of N could be used to decide if an element of Q is trivial.

□

1.6 Remark. There is another way of using Rips' construction (or the adapted Rips construction of Theorem 1.3) in order to produce incoherent groups. A theorem of Bieri [5] states that a finitely presented normal subgroup of a finitely presented group of cohomological dimension 2 is either free or of finite index. Since (torsion-free) small-cancellation groups (or negatively curved 2-complexes respectively) are of cohomological dimension 2 [9], one may apply Bieri's theorem to the group G . It follows that if Q is infinite, then either N is free or N is not finitely presented and so G is incoherent.

In the case of Rips' construction, N is generated by exactly 2 generators so any (non-trivial) relation shows that N is not free, and such a relation is supplied easily. In the case of the adapted Rips construction of Theorem 1.3, it is not immediately obvious that in general N is not free, though this is easy to check in special cases.

PART 2: A RECIPE FOR INCOHERENT FINITELY-PRESENTED GROUPS

Consider the finite presentation

$$\langle a_1, \dots, a_m, t \mid R = 1, a_1^t = W_1, \dots, a_m^t = W_m \rangle$$

where W_i and R are words in $a_1^{\pm 1} \dots a_m^{\pm 1}$. Let X denote the standard 2-complex associated with this presentation. Let $G = \pi_1(X)$ and let $A \subset G$ denote the subgroup generated by $\{a_1 \dots a_m\}$.

2.1 Theorem: Aspherical $\Rightarrow$ Incoherent. *If X is aspherical and R is not the empty word, then $\text{rank}(H_2(A)) = \infty$ and so A is not finitely presented.*

Proof. Let $\hat{X}$ denote the cover of X corresponding to the subgroup $A \subset G$. Let $B \subset X$ denote the subcomplex consisting of the 0-cell and the a_j labelled 1-cells. Let $\hat{B} \subset \hat{X}$ denote the lift of B at the basepoint.

As X is a $K(\pi, 1)$ space for G , $\hat{X}$ is a $K(\pi, 1)$ space for A , so $H_2(A) = H_2(\hat{X})$.

Denote by r the 2-cell of X associated with the relation $R = 1$.

FIGURE 2. The sequence of diagrams in the above figure are meant to suggest D_0 , D_1 and D_2 from left to right. The shaded triangle in the center of each of the diagrams corresponds to an r 2-cell of X . The remaining 2-cells correspond to relations of the form: $a_i^t = W_i$. The edges with a black arrow correspond to t letters. The basepoint of each diagram is the large grey vertex at its lower-left corner.

For each integer $n \geq 0$ we will form a combinatorial disc-diagram D_n mapped into X . Each disc-diagram in our sequence will have the property that ∂D_n is mapped into B .

This sequence of disc-diagrams corresponds to a sequence of relations among the generators of A . The idea of the proof is to use the sequence of 2-chains corresponding to the lifts of the D_n to show that $H_2(\hat{X})$ has infinite rank.

D_0 is the obvious disc-diagram corresponding to the 2-cell r . For each n , D_{n+1} is an extension of D_n by an annulus which is mapped into $X - r$. This annulus has width one and is composed of 2-cells corresponding to the $a_i^t = W_i$ relations joined along t -edges.

To be more explicit, let M denote the free monoid on $\{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$. The map $\{a_i \mapsto W_i, a_i^{-1} \mapsto (W_i)^{-1} \mid 1 \leq i \leq m\}$ induces an endomorphism $\tau : M \rightarrow M$. In this notation the label of ∂D_n beginning at the basepoint of D_n is $\tau^n(R) \in M$.

For each n we orient D_n and lift it to $\hat{D}_n \subset \hat{X}$, thus obtaining in an obvious manner an element $d_n \in C_2(\hat{X})$ of the 2-chain-complex of $\hat{X}$.

Since the boundary of each D_n diagram is mapped to $B \subset X$, it is true that the boundary of each $\hat{D}_n$ is mapped to $\hat{B}$. Therefore for each n we have

$$(2) \quad \partial d_n \in C_1(\hat{B}) \subset C_1(\hat{X}).$$

Observe that the image of $t \in G$ has infinite order in the natural homomorphism $G \rightarrow G/\langle\langle A \rangle\rangle \cong Z$. It follows that for $n \neq 1$, $t^n \notin A$, so the paths in $\hat{X}$ beginning at the basepoint and labelled by t^n all have distinct endpoints.

Note that the r -labelled 2-cell at the center of each disc-diagram D_n is connected to the basepoint of D_n by a path labeled by t^n .

It follows that the r -labelled 2-cells at the centers of distinct D_n diagrams lift to distinct r -labelled 2-cells in $\hat{X}$.

It is then easy to see that the set of 2-chains $\{d_n \mid n \geq 0\} \subset C_2(\hat{X})$ forms a basis for the submodule that it generates.

It follows from equation (2) that ∂ maps this infinite rank submodule of $C_2(\hat{X})$ to the finite rank (actually rank m) submodule $C_1(\hat{B}) \subset C_1(\hat{X})$. The kernel of

∂ restricted to this infinite rank submodule must be of infinite rank. Therefore $H_2(\hat{X})$ must be of infinite rank as well. $\square$

2.2 Example: An incoherent small-cancellation group. It is easy to construct examples of presentations with the form of Theorem 2.1 which are small-cancellation presentations (without torsion), and therefore aspherical [9]. For instance,

$$\langle a, b, t \mid a^1 b^1 a^2 b^2 \dots a^{30} b^{30} = 1, a^t = ab^{10} ab^{11} \dots ab^{30}, b^t = ba^{10} ba^{11} \dots ba^{30} \rangle.$$

2.3 Example: An incoherent 2-relator group.

$$\langle a, b, t \mid a^t = a^{10} ba^{10}, b^t = b^{10} ab^{10}, a^9 b^2 a^8 b^3 a^7 b^4 a^6 b^5 a^5 b^6 a^4 b^7 a^3 b^8 a^2 b^9 = 1 \rangle.$$

The presentation above satisfies the $C(6)$ small cancellation condition and so is aspherical. A 2-relator presentation may be obtained by using the substitution $b = a^{-10} a^t a^{-10}$, followed by a Tietze transformation eliminating the generator b and the relator $a^t = a^{10} ba^{10}$. Substantially less complicated incoherent 2-relator examples could be formed using the same idea, but proving that the presentation in the first step is aspherical is more difficult.

2.4 Example: Incoherent negatively curved. Here is an example of a negatively curved 2-complex which is of the desired form. Metrize each 2-cell of the standard complex of

$$\langle a_1, \dots, a_7, t \mid a_7 a_6 a_5 a_4 a_3 a_2 a_1, a_i^t = a_i a_i a_{i+1} a_{i+3} \text{ (coefficients mod 7)} \rangle$$

as a regular hyperbolic heptagon with angles $\frac{2\pi}{3}$.

One must only check that the presentation has no pieces of length 2 in the sense of Lyndon (see [9]). This means that no subword of length 2 appears more than once among the cyclic conjugates of the relators and their inverses.

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REFERENCES

- [1] G. Baumslag, *Some Problems in One-Relator Groups*, in: Proc. Conf. Canberra 1973, Lecture notes in Math Vol. 372 (1973), 75-81. MR **51**:717
- [2] G. Baumslag, M. R. Bridson, C. F. Miller III, H. Short, *Subgroups of Automatic Groups*, pre-print.
- [3] G. Baumslag, C. F. Miller III, H. Short, *Unsolvable problems about Small Cancellation and Word Hyperbolic Groups*, Bull. London Math. Soc. **26** (1994), 97-101. MR **94i**:20053
- [4] G. Baumslag and J. E. Roseblade, *Subgroups of Direct Product of Two Free Groups*, J. London Math. Soc. (2) **30** (1984), 44-52. MR **86d**:20028
- [5] R. Bieri, *Homological dimension of discrete groups; Queen Mary College Math. Notes*, Queen Mary College, London, 1976. MR **57**:6224
- [6] M. R. Bridson and A. Haefliger, *Metric spaces of nonpositive curvature*, to appear.
- [7] S. M. Gersten, *Questions on Geometric Group Theory for the Max Dehn Seminar*, available by ftp at: ftp.math.utah.edu /u/ma/gersten/MaxDehnSeminar (1995).
- [8] M. Gromov, *Hyperbolic Groups*, in: Essays in Group Theory (S.M. Gersten ed.) (1987). MR **89e**:20070
- [9] R. C. Lyndon, and P. E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, 1977. MR **58**:28182

- [10] E. Rips, *Subgroups of Small Cancellation Groups*, Bull. London Math. Soc. (1982), 45-47. MR **83c**:20049
- [11] G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, J. London Math. Soc. **6** (1973), 437-440. MR **52**:1660
- [12] J. P. Serre, *Problems*, in: Proc. Conf. Canberra 1973, Lecture notes in Math Vol. 372 (1974), 734-735. MR **51**:656

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