REMARKS ON THE RESULTS BY KOSKELA CONCERNING THE RADIAL UNIQUENESS FOR SOBOLEV FUNCTIONS

YOSHIHIRO MIZUTA

(Communicated by Albert Baernstein II)

ABSTRACT. In this note we aim to complete the results by Koskela concerning the radial uniqueness for Sobolev functions.

Let φ be a positive nonincreasing function on the interval $(0, \infty)$, and let $\mathbf{B}$ denote the unit ball of R^n . Consider a p -precise function u on $\mathbf{B}$ such that

$$\int_{U(\varepsilon)} |\nabla u(x)|^p dx \leq \varepsilon^p \varphi(\varepsilon) \quad \text{for any } \varepsilon > 0,$$

where $U(\varepsilon) = \{x \in \mathbf{B} : |u(x)| < \varepsilon\}$. We give conditions on φ which assure that $u = 0$ whenever u has vanishing fine boundary limits on a set of positive p -capacity.

We are also concerned with the sharpness.

1. STATEMENT OF RESULTS

For $1 < p < \infty$ and an open set $G \subset R^n$, we denote by $W^{1,p}(G)$ the Sobolev space of all functions u on G such that

$$\int_G |\nabla u(x)|^p dx < \infty,$$

where ∇ denotes the gradient. For a set $E \subset G$, we define

$$C_p(E; G) = \inf \int_G |\nabla u(x)|^p dx,$$

where the infimum is taken over all $u \in W^{1,p}(R^n)$ such that $u = 0$ outside G and

$$\int |x - y|^{1-n} |\nabla u(y)| dy \geq 1 \quad \text{for every } x \in E$$

(see [2], [7] and [9]). We write $C_p(E) = 0$ if $C_p(E \cap G; G) = 0$ for every bounded open set G .

Let $\mathbf{B}$ denote the unit ball in R^n . For each $u \in W^{1,p}(\mathbf{B})$, we can find an extension $u^* \in W^{1,p}(R^n)$ with compact support such that

$$u = u^* \quad \text{for almost every } x \in \mathbf{B}$$

Received by the editors September 18, 1996.

1991 *Mathematics Subject Classification*. Primary 31B25, 31B15, 46E35.

Key words and phrases. p -precise functions, Sobolev functions, capacity, fine boundary limits.

and u^* is p -precise in the sense of Ziemer [10]. In view of [4], [5], we note further that u^* can be written as

$$u^*(x) = c \int \frac{x-y}{|x-y|^n} \cdot \nabla u^*(y) dy$$

with a constant c , where “ $\cdot$ ” denotes the usual inner product and the integrals are absolutely convergent p -q.e. (quasi everywhere) on R^n , that is, for every $x \in R^n - E$ with $C_p(E) = 0$. This implies that $u \in W^{1,p}(\mathbf{B})$ has a fine boundary limit $u^*(\xi)$ for p -q.e. $\xi \in \partial\mathbf{B}$ (see [3], [7] and [11]).

Our aim in this note is to show the following theorem.

Theorem 1. *Let φ be a positive nonincreasing function on the interval $(0, \infty)$ satisfying*

$$(1) \quad A^{-1}\varphi(r) \leq \varphi(r^2) \leq A\varphi(r) \quad \text{for every } r > 0$$

with a positive constant A and

$$(2) \quad \int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr = \infty, \quad 1 < p < \infty.$$

Let u be a Sobolev function in $W^{1,p}(\mathbf{B})$ such that

$$(3) \quad \int_{U(\varepsilon)} |\nabla u(x)|^p dx \leq \varepsilon^p \varphi(\varepsilon) \quad \text{for any } \varepsilon > 0,$$

where $U(\varepsilon) = \{x \in \mathbf{B} : |u(x)| < \varepsilon\}$. If there exists $E \subset \partial\mathbf{B}$ such that $C_p(E) > 0$ and the p -precise extension u^ vanishes on E , then $u = 0$.*

Remark. P. Koskela [1] has recently treated the case where

$$\begin{aligned} \varphi(r) = & [\log(1/r)]^{p-1}, \quad [\log(1/r)]^{p-1} [\log \log(1/r)]^{p-1}, \\ & [\log(1/r)]^{p-1} [\log \log(1/r)]^{p-1} [\log \log \log(1/r)]^{p-1}, \dots \end{aligned}$$

for small r ; these functions clearly satisfy (2).

Next we show that our result is sharp in the following sense.

Theorem 2. *Let φ be a positive nonincreasing function on the interval $(0, \infty)$ satisfying (1) and*

$$(4) \quad \int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty, \quad 1 < p < \infty.$$

Then there exists a continuous function u on R^n such that $u \in W^{1,p}(R^n)$, u satisfies (3), $u > 0$ on $\mathbf{B}$ and $u = 0$ outside $\mathbf{B}$.

2. PROOF OF THEOREM 1

We use the notation $B(x, r)$ to denote the open ball centered at x with radius r ; hence $\mathbf{B} = B(0, 1)$.

Let us begin with the following lemma, which is an easy consequence of the definition of C_p .

Lemma 1. *Let $1 < p < \infty$. For $f \in L^p(R^n)$, set*

$$E_f = \{x \in R^n : \int_{B(x,1)} |x-y|^{1-n} |f(y)| dy = \infty\}.$$

Then $C_p(E_f) = 0$ (see e.g. [7, Proposition 5.1.1 and Theorem 6.8.2]).

Next we prepare the following technical lemma.

Lemma 2. *Let φ be as in Theorem 1. Then there exists a positive nondecreasing function a on $(0, \infty)$ satisfying (1) and such that*

$$(5) \quad \int_0^1 a(r)r^{-1}dr = \infty$$

and

$$(6) \quad \int_0^1 a(r)^p \varphi(r)r^{-1}dr < \infty.$$

In fact, it suffices to consider

$$a(r) = [\varphi(r)]^{-1/(p-1)} \left(\int_r^2 [\varphi(t)]^{-1/(p-1)} t^{-1} dt \right)^{-1}$$

for $r \leq 1$; set $a(r) = a(1)$ for $r > 1$.

Throughout this note, let M denote various constants independent of the variables in question.

Now we are ready to prove Theorem 1. For this purpose, we may assume that $u \geq 0$ a.e. on $\mathbf{B}$. We suppose that $\{x \in \mathbf{B} : u(x) > 0\}$ has positive measure, and obtain a contradiction.

For functions f and g , define

$$f^+(x) = \max\{f(x), 0\}$$

and

$$f \wedge g(x) = \min\{f(x), g(x)\}.$$

For each positive integer j , set

$$u_j(x) = 2^j[(u - 2^{-j})^+ \wedge 2^{-j}]$$

and

$$v = \sum_{j=1}^{\infty} a_j u_j \quad \text{with } a_j = a(2^{-j})$$

for the function a in Lemma 2. Note that

$$|\nabla u_j| = 2^j |\nabla u| \quad \text{a.e. on } G_j = \{x : 2^{-j} < u(x) < 2^{-j+1}\}$$

and

$$|\nabla u_j| = 0 \quad \text{a.e. outside } G_j.$$

Hence it follows from (3), (1) and (6) that

$$\begin{aligned} \int |\nabla v|^p dx &= \sum_{j=1}^{\infty} [a_j 2^j]^p \int_{G_j} |\nabla u|^p dx \\ &\leq \sum_{j=1}^{\infty} [a_j 2^j]^p [2^{-j+1}]^p \varphi(2^{-j+1}) \\ &\leq M \sum_{j=1}^{\infty} a_j^p \varphi(2^{-j}) \\ &\leq M \int_0^1 a(r)^p \varphi(r)r^{-1}dr < \infty. \end{aligned}$$

Consider p -precise extensions $u^*, v^* \in W^{1,p}(R^n)$ with compact support such that

$$u = u^* \quad \text{and} \quad v = v^*$$

for almost every $x \in \mathbf{B}$. If $\xi \in \partial\mathbf{B}$ and

$$\int |\xi - y|^{1-n} |\nabla u^*(y)| dy < \infty,$$

then $u^*(x + r(\xi - x))$ is absolutely continuous on $(0, \infty)$ for almost every $x \in \mathbf{B}$ (see [4, p.725] and [8]).

If $u^*(x) > 0$, $u^*(\xi) = 0$ and $f(r) = v^*(x + r(\xi - x))$ is absolutely continuous on $[0, 1]$, then the line $L(x, \xi) = \{x + r(\xi - x) : r \in [0, 1]\}$ meets with $G_j = \{x \in \mathbf{B} : 2^{-j} < u^*(x) < 2^{-j+1}\}$ for large j , say $j \geq j_0$, so that

$$\int_0^1 |f'(r)| dr \geq \sum_{j \geq j_0} a_j \geq M \int_0^{2^{-j_0}} a(r) r^{-1} dr = \infty$$

because of (5). This implies that

$$\int_{\mathbf{B}} |\xi - y|^{1-n} |\nabla v^*(y)| dy = \infty,$$

which gives a contradiction by Lemma 1, since $C_p(E) > 0$ and v^* is p -precise on R^n .

3. PROOF OF THEOREM 2

Suppose

$$\int_0^1 [\varphi(r)]^{-1/(p-1)} r^{-1} dr < \infty$$

and set

$$r = f(t) \equiv A \int_0^t [\varphi(s)]^{-1/(p-1)} s^{-1} ds$$

for a positive constant A . Note here that if $b > 0$, then

$$(7) \quad s^b \varphi(s) \leq M(b) t^b \varphi(t) \quad \text{whenever } t > s > 0$$

with a positive constant $M(b)$ (see e.g. [6, ($\varphi 5$)]).

We now consider the inverse function $t = f^{-1}(r)$ and define

$$u(x) = f^{-1}(1 - |x|)$$

for $x \in \mathbf{B}$; define $u = 0$ outside $\mathbf{B}$. In what follows we determine A for which u satisfies (3).

For small $\varepsilon > 0$, find $\delta > 0$ such that

$$\varepsilon = f^{-1}(\delta).$$

Then we have by (7)

$$\begin{aligned}
 \int_{\{x \in \mathbf{B}: |u(x)| < \varepsilon\}} |\nabla u(x)|^p dx &\leq M \int_0^\delta |(f^{-1})'(r)|^p dr \\
 &= M \int_0^\varepsilon \left| \frac{dr}{dt} \right|^{-(p-1)} dt \\
 &= A^{1-p} M \int_0^\varepsilon t^{p-1} \varphi(t) dt \\
 &\leq A^{1-p} M \varepsilon^p \varphi(\varepsilon).
 \end{aligned}$$

Hence it suffices to take A such that $A^{1-p} M = 1$.

REFERENCES

- [1] P. Koskela, A radial uniqueness theorem for Sobolev functions, *Bull. London Math. Soc.* 27 (1995), 460-466. MR **96e**:31010
- [2] N. G. Meyers, A theory of capacities for potentials in Lebesgue classes, *Math. Scand.* 26 (1970), 255-292. MR **43**:3474
- [3] N. G. Meyers, Taylor expansion of Bessel potentials, *Indiana Univ. Math. J.* 23 (1974), 1043-1049. MR **50**:980
- [4] Y. Mizuta, Existence of various boundary limits of Beppo Levi functions of higher order, *Hiroshima Math. J.* 9 (1979), 717-745. MR **81d**:31013
- [5] Y. Mizuta, Boundary behavior of p -precise functions on a half space of R^n , *Hiroshima Math. J.* 18 (1988), 73-94. MR **89d**:31014
- [6] Y. Mizuta, Continuity properties of potentials and Beppo-Levi-Deny functions, *Hiroshima Math. J.* 23 (1993), 79-153. MR **94d**:31005
- [7] Y. Mizuta, *Potential theory in Euclidean spaces*, Gakkōtōsyō, Tokyo, 1996. CMP 97:06
- [8] M. Ohtsuka, *Extremal length and precise functions in 3-space*, Lecture Notes, Hiroshima University, 1973.
- [9] Yu. G. Reshetnyak, The concept of capacity in the theory of functions with generalized derivatives, *Siberian Math. J.* 10 (1969), 818-842. MR **43**:2234
- [10] W. P. Ziemer, Extremal length as a capacity, *Michigan Math. J.* 17 (1969), 117-128. MR **42**:3299
- [11] W. P. Ziemer, *Weakly differentiable functions*, Springer-Verlag, New York, 1989. MR **91e**:46046

THE DIVISION OF MATHEMATICAL AND INFORMATION SCIENCES, FACULTY OF INTEGRATED ARTS AND SCIENCES, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA 739, JAPAN

E-mail address: mizuta@mis.hiroshima-u.ac.jp