PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2725–2732
S 0002-9939(98)04345-7

ON THE RELATIVE COMMUTANTS OF SUBFACTORS

M. KHOSHKAM AND B. MASHOOD

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let $A \subset B$ be factors generated by a periodic tower $A_n \subset B_n$ of finite dimensional C^*-algebras. We prove that for sufficiently large n, $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_n \cap B_n$.

INTRODUCTION

This paper is mainly concerned with the structure of the relative commutants $A' \cap B$ of a pair $A \subset B$ of type II_1 factors, generated by an infinite ladder $A_n \subset B_n$ of finite dimensional C^*-algebras. If $A_n \subset B_n$ is obtained by iteration of the basic construction on $B_0 \subset B_1$, then it is a well known result of A. Ocneanu that $A' \cap B \subset A'_n \cap B_n$; however, this is not true in general. For example, the tower of higher relative commutants associated with a pair of non-amenable subfactors does not satisfy this property (cf. [8]). We prove a weaker version of this property for general periodic towers . The periodicity is needed to ensure that the Jones index $[A : B]$ is finite. But we do not assume $A_n \subset B_n$ to be a tower of commuting squares. Specifically, we prove that $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_n \cap B_n$ for sufficiently large n. This will have several interesting consequences, one being a generalization of (Theorem 1.7, [9]). Our main tool is the perturbation technique developed by E. Christensen in [1]. Given $\epsilon > 0$, we prove that for sufficiently large n, $A' \cap B \overset{\epsilon}{\subset} A'_n \cap B_n$; then we are in a situation to use perturbation theory and get the desired isomorphism. The last section of the paper deals with towers of commuting squares. In particular, we obtain a sufficient condition for a tower $A_n \subset B_n$ to be a tower of commuting squares.

§1. NOTATIONS AND PRELIMINARIES

Let B be a type II_1 factor and let tr be the faithful, normal and normalized trace on B. Denote by $L^2(B, tr)$ the Hilbert space closure of B under the norm given by the inner product $\langle x, y\rangle = tr(y^*x)$. Then, B acts on $L^2(B, tr)$ by left multiplication, and the identity of B is a cyclic and separating vector for B denoted by ξ_0. The involution $x \to x^*$ extends to a conjugate linear isometry on $L^2(B, tr)$ denoted by J_B. If A is a von Neumann subalgebra of B, let E_A be the conditional expectation

Received by the editors February 11, 1997.
1991 *Mathematics Subject Classification.* Primary 46L37.
Key words and phrases. Subfactors, von Neumann algebras, Jones index, commutant, commuting squares.

from B onto A associated with the trace, so that $tr(E_A(x)) = tr(x)$ for every $x \in B$. The extension of E_A to $L^2(B, tr)$, denoted by e_A, is the orthogonal projection of $L^2(B,\ tr)$ onto the closure of A regarded as a subspace of $L^2(B, tr)$. Then, $\langle B,\ e_A\rangle$ the von Neumann algebra generated by B and e_A is called the basic construction. The following facts from [3] are frequently used in this the paper and are listed below for convenience.

(a) If $x \in B$, then $x \in A \Leftrightarrow xe_B = e_B x$,
(b) if $x \in B$, then $e_A x e_A = E_N(x)e_A$,
(c) $J_B\langle B, e_A\rangle J_B = A'$,
(d) the map $x \to xe_A$ is an isomorphism of A onto Ae_A,
(e) $e_A\langle B, e_A\rangle e_A = Ae_A$

The index of A in B, denoted by $[B : A]$, is defined to be $(tr_{A'}(e_A))^{-1}$ if A' is finite and to be infinite otherwise (cf. [3]). It is a remarkable result of Jones that $[B : A] \in \{4\cos^2 \frac{\pi}{n} : n = 3, 4, \cdots\} \cup [4, \infty)$, $A' \cap B = \mathbf{C}$ if $[B : A] < 4$, and $A' \cap B$ is finite dimensional if $[B : A] < \infty$.

If $A \subset B$ are finite dimensional C^*-algebras with $\{e_i : i = 1, 2, \cdots, k\}$ and $\{f_j : j = 1, \cdots, l\}$ the sets of minimal central projections of A and B respectively, then the inclusion matrix $T_A^B = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 0, & \text{if } e_i f_j = 0; \\ \frac{\dim B_{e_i f_j}}{\dim A_{e_i f_j}} & \text{if } e_i f_j \neq 0. \end{cases}$$

The matrix T_A^B is unique up to permutations of the minimal central projections. It is an important fact that if $\langle B, e_A\rangle$ is the basic construction, then $\langle B, e_A\rangle$ is also finite dimensional and $T_B^{\langle B, e_A\rangle} = (T_A^B)^t$. The inclusion $A \subset B$ may also be described by a bipartite graph called the Bratteli diagram of the pair $A \subset B$, with the blocks of A and B forming the vertices of the graph and the edges being the multiplicities of subfactors of A in the subfactors of B (cf.[2],[3]). Increasing sequences of finite dimensional C^*-algebras (A_n), (B_n), with $A_n \subset B_n$, are said to constitute a periodic tower of algebras if for some h and all n we have:

$$T_{B_{n+h}}^{B_{n+h+1}} = T_{B_n}^{B_{n+1}},\ T_{A_{n+h}}^{A_{n+1+h}} = T_{A_n}^{A_{n+1}},\ T_{A_n}^{B_n} = T_{A_{n+h}}^{B_{n+h}}.$$

If there exists a faithful trace on $\bigcup B_n$, then $B = \overline{\bigcup B_n}$ and $A = \overline{\bigcup A_n}$ are the closures with respect to the weak topology induced by the trace. We say that $A \subset B$ is generated by the tower $A_n \subset B_n$. If the inclusion matrices are indecomposable and the tower is periodic, then A and B are factors and $[A : B] < \infty$. A diagram

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ C & \subset & D \end{array}$$

of finite dimensional C^*-algebras is called a commuting square with respect to a trace , tr, on B if

$$\begin{array}{ccc} A & \xleftarrow{E_A} & B \\ \cup & & \cup \\ C & \xleftarrow{E_C} & D \end{array}$$

is a commutative diagram, i.e., for each $x \in D$, $E_A(x) \in C$. The expectations E_A and E_C correspond to the trace, and we assume tr is a Markov trace.

§2. Main results

Near inclusions $A \overset{\delta}{\subset} B$ of pairs of von Neumann subalgebras of a type II_1 von Neumann algebra C were studied by E. Christensen in [1]. The relation $A \overset{\delta}{\subset} B$ means that for every $x \in A_1$, the unit ball of A, there exists an element $y \in B$ such that $||x - y||_2^{tr} < \delta$, where tr is the faithful, normal, and normalized trace on C. If E_B denotes the canonical trace preserving conditional expectation from C onto B, then

$$A \overset{\delta}{\subset} B \Longleftrightarrow ||x - E_B(x)||_2^{tr} < \delta \ \forall x \in A_1. \tag{1}$$

As we are considering pairs of factors generated by finite dimensional algebras, we only need to consider inclusions $A \overset{\delta}{\subset} B$ with B finite dimensional. We need to modify several perturbations lemmas from [1]. However, the main ideas remain the same. Given $\delta > 0$, the following function is frequently needed:

$$\gamma(\delta) = 2^{\frac{1}{4}}\delta^{\frac{1}{2}}(1 - 2^{\frac{1}{4}}\delta^{\frac{1}{2}})^{-1}. \tag{2}$$

Lemma 2.1. *Let B be a finite dimensional subalgebra of a type II_1 von Neumann algebra C. If tr is the faithful, normal, and normalized trace on B, then there exists a semifinite trace, tr', on $\langle C, e_B\rangle$ such that*

$$\begin{aligned} tr'(xe_B) &= tr(x) \ \forall x \in C, \\ tr'(e_B) &= 1. \end{aligned} \tag{3}$$

Proof. Let $e_1, e_2, \cdots, e_m$ be the minimal central projections of B. By [3, Proposition 3.1.5], $f_k = J_C e_i J_C, 1 \le k \le m$, are the minimal central projections of the semifinite von Neumann algebra $\langle C, e_B\rangle$. Hence, there exists a unique faithful normal semifinite trace, tr', on $\langle C, e_B\rangle$ such that $tr'(f_k e_B) = tr(e_k)$. Using the $*$-isomorphism $xf_k e_B \to E_B(x)e_k$ from $\langle C, e_B\rangle f_k e_B$ onto Be_k (cf. [3]) and the uniqueness of the trace on the factor Be_k, we have that $tr'(xf_k e_B) = tr(E_B(x)e_k)$ for all $x \in C$. Given $x \in C$, we have

$$\begin{aligned} tr'(xe_B) &= tr'(\sum_{k=1}^m xf_k e_B) \\ &= \sum_{k=1}^m tr'(xf_k e_B) \\ &= \sum_{k=1}^m tr(E_B(x)e_k) \\ &= tr(E_B(x)\sum_{k=1}^m e_k) = tr(x). \end{aligned}$$

Let $x = 1$ in the above to get $tr'(e_B) = 1$. □

Lemma 2.2. *Let B, C, and $e_1, e_2, \cdots, e_m$ be as above. Suppose that A is a von Neumann subalgebra of C such that $A \overset{\delta}{\subset} B$ with $\delta < \frac{1}{\sqrt{2}}$ and such that for every minimal projection $q \in B$, $\gamma^2(\delta) < tr(q)$. Then, there exists a $*$-homomorphism φ from A into B. Moreover, for each x in the unit ball of A we have:*

$$||\varphi(x) - x|| < 26\gamma(\delta) + \delta, \qquad \varphi(I) = I. \tag{4}$$

Proof. If $u \in A$ is unitary, then

$$tr(I - E_B(u^*)E_B(u)) = ||u - E_B(u)||_2^{tr} < \delta^2.$$

If tr' is as in Lemma 2.1, then we have

$$\begin{aligned} tr'((e_B - u^*e_Bu)^2) &= tr'(e_B + u^*e_Bu - u^*e_Bue_B - e_Bu^*e_Bu) \\ &= 2tr(e_B(1 - E_B(u^*)E_B(u)) \\ &= 2tr(I - E_B(u^*)E_B(u)) \\ &< 2\delta^2. \end{aligned}$$

As in [1, Theorem 3.1], we can find an element k of minimal trace norm $|| \; ||_2^{tr'}$ in the ultrastrong closed convex hull of the set $\{u^*e_Bu : u \text{ unitary in } B\}$ such that $k \in A' \cap \langle A, e_B \rangle$ and

$$0 \leq k \leq I, \quad ||k - e_B||_2^{tr'} < 2^{\frac{1}{2}}\delta.$$

Since $tr'(e_B) = 1$, by the discussion preceding [1, Lemma 4.1] there exists a projection $q \in A' \cap \langle A, e_B \rangle$ such that

$$||q - e_B||_2^{tr'} \leq \gamma(\delta) \quad \text{and} \quad |1 - tr'(q)| < \gamma(\delta)^2.$$

Let $\{f_1, ..., f_m\}$ be the set of minimal central projection of $\langle C, e_B \rangle$ (see Lemma 2.1). Then,

$$\begin{aligned} |tr'(qf_n) - tr'(e_Bf_n)| &= |tr'((q - e_B)f_n))| \\ &\leq |tr'(q - e_B)| \\ &= |1 - tr'(q)| \\ &< \gamma(\delta)^2 \; . \end{aligned}$$

Since the factor $\langle C, e_B \rangle f_k$ is isomorphic to Be_k and $tr(p) > \gamma(\delta)^2$ for every projection $p \in B$, the above inequality implies that $tr'(qf_k) = tr'(e_Bf_k)$. Hence, $qf_k \sim e_Bf_k$. Let $v_k \in \langle C, e_B \rangle f_k$ be a partial isometry such that $v_kv_k^* = qf_k$ and $v_k^*v_k = e_Bf_k$. Then, $q \sim e_B$ via the partial isometry $v = v_1 + v_2 + \cdots + v_m$. It is easy to check that $m \to v^*mv$ is a homomorphism from A into $\langle C, e_B \rangle e_B = Be_B$. Let φ be the composition of this with the canonical identification of Be_B with B. If x is in the unit ball of A, then by [2, Lemma 4.1], $||\varphi(x) - E_A(x)|| < 26\gamma(\delta)$. Since $||x - E_A(x)|| < \delta$, the inequality (4) follows. □

Proposition 2.3. *Let $A \subset B$ be factors generated by the tower $A_n \subset B_n$ such that $[B : A] < \infty$. Then, given $\epsilon > 0$, there exists $N > 0$ such that:*

$$A' \cap B \overset{\epsilon}{\subset} A'_n \cap B_n \, \forall n > N.$$

Proof. Since $B = \overline{\bigcup B_n}$, given x in the unit ball of $A' \cap B$ there exist an integer n and $y \in B_n$ such that $||x - y||_2^{tr} < \frac{\epsilon}{3}$. Hence, $\| \; x - E_{B_n}(x) \; \|_2^{tr} < \frac{2\epsilon}{3}$. Since $[B : A] < \infty$ by [3, Corollary 2.2.3] $A' \cap B$ is finite dimensional, and we may use a standard compactness argument and find a positive N such that

$$\| \; x - E_{B_n}(x) \; \|_2^{tr} < \epsilon \, \forall x \in (A' \cap B)_1 \quad \text{and} \; \forall n > N. \tag{6}$$

It is easy to see that if $x \in A' \cap B$, then $E_{B_n}(x) \in A'_n \cap B_n$ and hence $A' \cap B \overset{\epsilon}{\subset} A'_n \cap B_n$ by (6) and (1). □

Lemma 2.4. *Let $A \subset B$ be factors generated by $A_n \subset B_n$. If $\limsup ||T_{A_n}^{B_n}|| \leq \infty$, then $[A : B] < \infty$.*

Proof. It follows from [5, Propositions 2.6 and 3.4] that the entropy $H(B, A) \leq \limsup ||T_{A_n}^{B_n}||$. If $A' \cap B$ had a completely non-atomic part, then as in the proof of [5, Theorem 4.4] we conclude that $H(B, A) = \infty$, which contradicts the earlier statement. Hence, $A' \cap B$ must be atomic and by [5, Theorem 4.4], we conclude that $[B : A] < \infty$. □

Corollary 2.5. *Let $A \subset B$ be generated by a periodic tower $A_n \subset B_n$. Then, given $\varepsilon > 0$, there exists N such that*

$$A' \cap B \overset{\varepsilon}{\subset} A_n' \cap B_n \, \forall n > N.$$

Proof. If $A_n \subset B_n$ is periodic, then the hypothesis of Lemma 2.4 is satisfied and hence $[A : B] < \infty$. Now Proposition 2.3 can be used. □

Theorem 2.6. *Let $A \subset B$ be factors generated by $A_n \subset B_n$ and let $\{e_n^k\}_k$ be the set of minimal central projections of A_n. Suppose that*

(a) *$[B : A] < \infty$,*
(b) *$\liminf_{k,n} (tr(e_n^k)) > 0$,*
(c) *there exists $r > 0$ such that for every n and every minimal projection $p \in A_n' \cap B_n$ we have $tr(p) > r$.*

Then $A' \cap B$ is $*$-isomorphic to a subalgebra of $(A_n' \cap B_n)_{e_n^k}$ for sufficiently large n.

Proof. Choose $d > 0$ such that $\liminf_{k,n} \; tr(e_n^k) > d > 0$ and choose $\epsilon > 0$ such that $\epsilon < \frac{1}{\sqrt{2}}$, $\gamma^2(\epsilon) < r$, and $\gamma(\epsilon) + \epsilon < tr(p)$ for every minimal projection p in the finite dimensional algebra $A' \cap B$. Next choose $\delta > 0$ such that $\frac{\delta}{d} < \epsilon$. By Proposition 2.3 there exists N such that $A' \cap B \overset{\delta}{\subset} A_n' \cap B_n$ when $n > N$. Let $\tilde{t}r(x) = \frac{tr(x)}{tr(e_n^k)}$ for $x \in (B_n)e_n^k$. Then $\tilde{t}r_r$ is the canonical normalized trace on $(B_n)_{e_n^k}$, and we obtain

$$(A' \cap B)_{e_n^k} \overset{\frac{\delta}{d}}{\subset} (A_n' \cap B_n)_{e_n^k} \;\; \forall k$$

where the above near inclusion is with respect to the $\tilde{t}r$ norm. Since, $\frac{\delta}{d} < \epsilon$ we have that

$$(A' \cap B)_{e_n^k} \overset{\epsilon}{\subset} (A_n' \cap B_n)_{e_n^k} \;\; \forall k, n > N.$$

By the choice of ϵ the hypothesis of Lemma 2.2 is satisfied, and hence there exists a $*$-homomorphism φ from $(A' \cap B)_{e_n^k}$ into $(A_n' \cap B_n)_{e_n^k}$. If $\varphi(x) = 0$ for some $x \in (A' \cap B)_{e_{k,n}}$, then $\varphi(p) = 0$ for some projection. By (4), $tr(p) < \gamma(\epsilon) + \epsilon$. This contradicts the choice of ϵ. Hence, φ must be one to one. Finally, as $e_n^k \in A$ and A is a factor, $(A' \cap B)_{e_n^k}$ is $*$-isomorphic to $A' \cap B$, and the proof is complete. □

Theorem 2.7. *Suppose that $A \subset B$ is generated by a periodic tower $A_n \subset B_n$ of finite dimensional C^*-algebras, then $A' \cap B$ is $*$-isomorphic to a subalgebra of $A_n' \cap B_n$.*

Proof. Since $A_n \subset B_n$ is periodic by Corollary 2.5, the index $[B : A]$ is finite. Conditions (b) and (c) of Theorem 2.6 are satisfied here because the inclusion matrices are periodic. We refer to [9] for a proof of this fact. Hence the result follows from Theorem 2.6. □

The following generalizes Theorem 1.7 of [9] in the sense that $A_n \subset B_n$ is not assumed to be a tower of commuting squares.

Corollary 2.8. *If $A \subset B$ and $A_n \subset B_n$ are as above and there exists $m > 0$ such that $(A'_n \cap B_n)_{e_{k,n}} = \mathbf{C}$ for some k and every $n > m$. Then $A' \cap B = \mathbf{C}$.*

Proof. Follows directly from Theorems 2.5 and 2.7. □

Corollary 2.9. *If $A \subset B$ and the tower $A_n \subset B_n$ are as before, and for a sequence (n_k) the inclusion matrix $T_{A_{n_k}}^{B_{n_k}}$ has a column (or a row) of exactly one nonzero entry equal to one, then $A' \cap B = \mathbf{C}$.*

Proof. Our hypothesis implies that for some l, $(A'_{n_k} \cap B_{n_k})_{e_{l,n_k}} = \mathbf{C}$ for all n_k. Now the result follows from Corollary 2.8. □

Proposition 2.10. *Let $A \subset B$, with $[B : A] < \infty$, be factors generated by a tower $A_n \subset B_n$ of commuting squares. Let $\{e_n^k\}$ and $\{f_n^l\}$ be the set of minimal central projections of A_n and B_n respectively and suppose that $\liminf_{k,n}(tr(e_n^k)) > 0$. Then, for sufficiently large n, $A' \cap B$ is $*$-isomorphic to a subalgebra of $A'_n \cap B_n$.*

Proof. By [5, Proposition 2.6] $[B : A] = \lim \lambda(B_n : A_n)^{-1}$, and by [6, Theorem 2.3] $\lambda(B_n, A_n)^{-1} = \max_l(\sum_k a_{kl}^2 tr(e_n^k)/tr(e_n^k f_n^l))$ where a_{kl}^2 is just the dimension of $(A'_n \cap B_n)e_n^k f_n^l$. Hence, there exists N such that for every $n > N$ we have

$$a_{kl}^2 tr(e_n^k)/tr(e_n^k f_n^l) \leq [B : A].$$

Equivalently, for $n > N$

$$\frac{tr(e_n^k f_n^l)}{a_{kl}^2} \geq \frac{tr(e_n^k)}{[B : A]}.$$

Now the left hand side of the above inequality is the trace of minimal projections in $(A'_n \cap B_n)e_n^k f_n^l$. Given that $\liminf tr(e_n^k) > 0$, it follows that condition (c) of Theorem 2.6 is satisfied here, and the result follows from that theorem. □

§3. Remarks on commuting squares

An important tool in construction and the study of subfactors is the concept of commuting squares (see Section 1 for the definition), developed my M. Pisner and S. Popa (cf. [5]), and used by many authors in relation with Jones index theory (cf. [4], [7], [9]). The construction and classification of subfactors are closely related to those of commuting squares. The results of this section are along these lines. If $B_2 = \langle B_1, e_1 \rangle$ is the basic construction on $A_1 \subset B_1$ and if $A_2 = \langle A_1, e_1 \rangle$, then the following is a commuting square:

$$\begin{array}{ccc} A_2 & \subset & B_2 \\ \cup & & \cup \\ A_1 & \subset & B_1 \end{array}$$

By iterating the above procedure we obtain the tower $A_n \subset B_n$ of commuting squares. Note that the algebra A_n contains the set $\{e_1, e_2, ..., e_{n-2}\}$ of projections corresponding to the basic construction. We want to show that the converse is also true in the following sense. Namely, let the tower $B_n \subset B_{n+1}$ be obtained by iterating the basic construction on the pair of finite dimensional C^*-algebras $B_1 \subset B_2$, and let $\{e_n\}$ be the corresponding Jones' projection. Suppose that $\{A_n\}$ is an increasing sequence of C^*-algebras with $A_n \subset B_n$ such that for $n \geq 3$, A_n

contains the set $\{e_1, e_2, ..., e_{n-2}\}$. Set $A = \overline{\bigcup A_n}$, $B = \overline{\bigcup B_n}$ and $\tilde{A}_n = A \cap B_n$, where the closures are in the sense of the weak operator topology. With these notations, we state the following.

Proposition 3.1. *$\tilde{A}_n \subset B_n$ is a tower of commuting squares. Moreover, if $[B : A] < \infty$, then the tower $\tilde{A}_n \subset B_n$ is periodic.*

Proof. We must show that if $y \in \tilde{A}_n$, then $E_{B_{n-1}}(y) \in \tilde{A}_{n-1}$. Since $e_n e_{n-1} e_n = \lambda e_n$ with $\lambda^{-1} = \|T_{B_1}^{B_2}\|$, we have that

$$\begin{aligned} e_n e_{n-1} y e_{n-1} e_n &= e_n E_{B_{n-1}}(y) e_{n-1} e_n \\ &= E_{B_{n-1}}(y) e_n e_{n-1} e_n \\ &= \lambda E_{B_{n-1}}(y) e_n. \end{aligned}$$

Hence, $e_n E_{B_{n-1}}(y) \in \tilde{A}_{n+2} \subset A$. It follows that $e_m E_{B_{n-1}}(y) \in A$ for all $m \geq n$. Hence, $e_n \vee e_{n+1} \vee \cdots \vee e_m E_{B_{n-1}}(y) \in A$ for $m > n$. Since $tr(1 - e_{n+1} \vee \cdots \vee e_m)$ converges to zero as m tends to infinity (cf. §4, [2]), the projections $e_n \vee \cdots \vee e_m$ converge to the identity as m tends to infinity. Hence, $E_{B_{n-1}}(y) \in A \cap B_{n-1} = \widetilde{A}_{n-1}$, proving the first statement. When $[B : A] < \infty$ the periodicity is a consequence of the fact that $[B : A] = \lim [B_n : A_n]$ (cf. [5]). □

We obtain the following corollary regarding intermediate subfactors.

Corollary 3.2. *Let $A_n \subset B_n$ be obtained by the basic construction and let C be a middle subfactor, $A \subset C \subset B$. Then there exists a periodic tower of commuting squares $A_n \subset C_n \subset B_n$ such that $C = \lim C_n$.*

Proof. Let $C_n = B_n \cap C$ and apply the previous lemma. □

The following theorem shows that for factors $A \subset B$ generated by a tower $A_n \subset B_n$ the existence of proper intermediate subfactors may be reduced to a finite dimensional problem.

Theorem 3.3. *Let $A \subset B$ be generated by the tower $A_n \subset B_n$ given by the basic construction. Then there exists a proper middle subfactor C if and only if, for some n, there exist C^*-algebras $C_n \subset C_{n+2}$ such that*

$$\begin{array}{ccccc} A_{n+2} & \subset & C_{n+2} & \subset & B_{n+2}, \\ \\ A_n & \subset & C_n & \subset & B_n, \end{array}$$

with $T_{C_n}^{B_n} = T_{C_{n+2}}^{B_{n+2}}$.

Proof. The only if part follows from the previous corollary. Now suppose that $C_n \subset C_{n+2}$ are as in the statement of the theorem. Since the identity is in $A_{n+2} \cap C_{n+2}$, for every minimal central projection q in C_{n+2}, qe_n is a nonzero projection. Then $e_n E_{B_n}(C_{n+1}) = e_n C_{n+1} e_n$ is a subset of $e_n (C_{n+2}) e_n = (C_n)_{e_n}$. The last equality holds because

$$(C_n)_{e_n} \subset (C_{n+2})_{e_n} \subset (B_{n+2})_{e_n} = (B_n)_{e_n}$$

and $T^{B_n}_{C_n} = T^{B_{n=2}}_{C_{n+2}}$. Therefore, $E_{B_n}(C_{n+1}) \subseteq C_n$. But $C_n \subseteq E_{B_n}(C_{n+2})$, whence $E_{B_n}(C_{n+1}) = C_n$. This shows that

$$\begin{array}{ccc} C_{n+1} & \subset & B_{n+1} \\ \cup & & \cup \\ C_n & \subset & B_n \end{array}$$

is a commuting square, and it is easy to see that $C_{n+2} = \langle C_{n+1}, e_n \rangle$. Then C can be constructed inductively. $\square$

References

[1] Christensen, E., *Subalgebras of a finite algebra.*, Math. Ann. **243** (1979), 17–29. MR **80i:**46051

[2] Goodman, F., De La Harpe, P.S., Jones, V.F.R., *Coxeter graphs and towers of algebras, Math. Sci. Res. Hist. Publ.*, Springer-Verlag **14** (1989). MR **91c:**46082

[3] Jones, V., *Index for subfactors*, Invent. Math. **72** (1983), 1–25. MR **84d:**46097

[4] Ocneanu, A., *Quantized groups, string algebras and Galois theory for von-Neumann algebras*, Operator algebras and applications, London Math. Soc. Lecture Notes ser. 136 119–172, Cambridge Univ. Press, 1988. MR **91k:**46068

[5] Pimsner, M., Popa, S., *Entropy and index for subfactors.*, Ann. Sci. Ecole Norm. Sup. **19** (1986), 57–106. MR **87m:**46120

[6] Pimsner, M., Popa, S., *Finite dimensional approximation for pair of algebras and obstruction for index*, J. Func. Anal. **98** (1991), 270–291. MR **92g:**46076

[7] Popa, S., *Classification of subfactors, reduction to commuting squares*, Invet. Math. **101** (1990), 19–43. MR **91h:**46109

[8] Popa, S., *Classification of amenable subfactors of type* II, Acta Math. **172** (1994), 163–255. MR **95f:**46105

[9] Wenzl, H., *Representation of Hecke algebras and subfactors*, Thesis, University of Pennsylvania.

Department of Mathematics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchawan, Canada S7N 5E6

E-mail address: khoshkam@math.usask.ca