SUFFICIENT CONDITIONS FOR ONE DOMAIN TO CONTAIN ANOTHER IN A SPACE OF CONSTANT CURVATURE

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ABSTRACT. As an application of the analogue of C-S. Chen's kinematic formula in the 3-dimensional space of constant curvature ϵ , that is, Euclidean space $\mathbb{R}^3$, 3-sphere S^3 , hyperbolic space $\mathbb{H}^3$ ($\epsilon = 0, +1, -1$, respectively), we obtain sufficient conditions for one domain to contain another domain in either an Euclidean space $\mathbb{R}^3$, or a 3-sphere S^3 or a hyperbolic space $\mathbb{H}^3$.

§1. INTRODUCTION

For a long time one is faced with the problem of fitting a rugby into a box. Generalizing this problem we ask: given two domains in a space $\mathbb{E}_\epsilon^n$ of constant curvature when can one be moved by an *isometry* “inside” the other. The answer should only depend on geometric invariants of the domains, such as volumes, surface areas and curvature integrals of the boundaries of domains. In the case of $\mathbb{E}^2$ sufficient conditions were given by Hadwiger [8], [9]. By using integral geometry, Grinberg, Ren and Zhou [12] were able to give a fairly simple sufficient condition for domains in a plane $\mathbb{E}_\epsilon^2$ of constant curvature, and one that is derived simultaneously for all ϵ . The key idea is to use the isoperimetric deficit which was first invoked by Ren [3] in the Euclidean plane. Higher dimensional analogues were found by Zhou [14], [15], [16], [17] and Zhang [5] in the Euclidean case. As one might expect the inequalities involve many invariants.

The purpose of this paper is to give sufficient conditions for one domain to contain another domain in a 3-dimensional space $\mathbb{E}_\epsilon^3$ of constant curvature. We use the analog of C-S. Chen's formula [4], [17], i.e., kinematic formula for curvature in the space $\mathbb{E}_\epsilon^3$ of constant curvature. In general the domains D_i and D_j are assumed to be C^2 -smooth. We also place an upper bound on the Euler-Poincaré characteristics of the intersections $D_i \cap gD_j$. The latter requirement can be avoided if one just considers smooth convex bodies D_i and D_j .

§2. ANALOGUE OF C-S. CHEN'S KINEMATIC FORMULA IN $\mathbb{E}_\epsilon^3$

Let M_i, M_j be two submanifolds in a homogeneous space G/H and I an invariant of the intersection submanifold $M_i \cap gM_j$. Let dg be suitably normalized kinematic

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density of G . Evaluating the integral

$$(1) \quad \int_{\{g \in G: M_i \cap gM_j \neq \emptyset\}} I(M_i \cap gM_j) dg$$

in terms of invariants of M_i and M_j we obtain the so-called **kinematic formula** [1], [2], [3], [6], [14], [17].

Let Σ be a closed surface of class C^2 in a 3-dimensional Euclidean space $\mathbb{R}^3$. Denote by R , H , respectively, the Gaussian curvature, the mean curvature of Σ . The total Gaussian curvature, the total square mean curvature are, respectively, defined by

$$(2) \quad \tilde{R} = \int_{\Sigma} R d\sigma, \quad \tilde{H}^2 = \int_{\Sigma} H^2 d\sigma.$$

Let M_i and M_j be two surfaces of class C^2 in $\mathbb{R}^3$. Assume that M_i is fixed, gM_j is moving under the isometry g . Let dg be the kinematic density for $\mathbb{R}^3$, so normalized that the measure of all positions about a point is $8\pi^2$. Let κ_{Γ_g} denote the curvature of the intersection curve $\Gamma_g = M_i \cap gM_j$. Denote by F_k the surface area, and $\tilde{R}_k$, $\tilde{H}_k^2$ the total Gaussian curvature, the total square mean curvature of M_k ($k = i, j$), respectively. Then we have the following C-S. Chen's kinematic formula [4], [17]

$$(3) \quad \int_G \left(\int_{M_i \cap gM_j} \kappa_{\Gamma_g}^2 ds \right) dg = 2\pi^3 \left\{ (3\tilde{H}_i^2 - \tilde{R}_i) F_j + (3\tilde{H}_j^2 - \tilde{R}_j) F_i \right\}.$$

Let $\mathbb{E}_{\epsilon}^3$ be the 3-dimensional simply connected space of constant curvature ϵ and IS_{ϵ}^3 the group of orientation preserving isometries of $\mathbb{E}_{\epsilon}^3$ with the normalized invariant density dg . Let $\Sigma \subset \mathbb{E}_{\epsilon}^3$ be a C^2 -smooth surface and W the **Weingarten transformation**. That is, if $\mathbf{n}$ is the unit normal to Σ then W is the linear map defined on the tangent bundle $T(\Sigma)$ by

$$WX := -\nabla_X \mathbf{n},$$

where ∇ is the Riemannian connection of $\mathbb{E}_{\epsilon}^3$ and $X \in T(\Sigma)$ a tangent field.

The principal curvature of Σ is defined as the eigenvalue of W . The *mean curvature* H_{Σ} and the *extrinsic Gauss curvature* R_{Σ}^e are, respectively, defined by

$$(4) \quad H_{\Sigma} = \frac{1}{2} \text{Tr}(W) = \frac{\lambda_1 + \lambda_2}{2}, \quad R_{\Sigma}^e = \det(W) = \lambda_1 \lambda_2,$$

where λ_1, λ_2 are eigenvalues of W .

By the *Gauss equation* in $\mathbb{E}_{\epsilon}^3$ the Gauss curvature R_{Σ} and the extrinsic Gauss curvature R_{Σ}^e are related by (see [7], [13])

$$(5) \quad R_{\Sigma} = R_{\Sigma}^e + \epsilon.$$

The transfer principle [6] tells us that *the form of kinematic formula in a homogeneous space G/H does not depend on the full group of isometries G , but only on the invariant theory of the isotropy subgroup H* . For example, this transfer principle allows us to obtain a new kinematic formula in the space $\mathbb{E}_{\epsilon}^n$ from a kinematic formula which has been proved in a Euclidean space $\mathbb{R}^n$ without changing invariants that are independent of the curvature ϵ of the space. Therefore by the

transfer principle Chen's kinematic formula holds in the 3-dimensional space $\mathbb{E}_\epsilon^3$ in the following form:

$$(6) \quad \int_{IS_\epsilon^3} \left(\int_{M_i \cap gM_j} \kappa_\Gamma^2 ds \right) dg = 2\pi^3 \left((3\tilde{H}_j^2 - \tilde{R}_j^e)F_i + (3\tilde{H}_i^2 - \tilde{R}_i^e)F_j \right),$$

where κ_Γ is the geodesic curvature of curve $\Gamma_g = M_i \cap gM_j$.

By use of the Gauss equation, the formula (6) can be rewritten in the following form¹:

$$(7) \quad \int_{IS_\epsilon^3} \left(\int_{M_i \cap gM_j} \kappa_\Gamma^2 ds \right) dg = 2\pi^3 \left((3\tilde{H}_j^2 - \tilde{R}_j)F_i + (3\tilde{H}_i^2 - \tilde{R}_i)F_j + 2\epsilon F_i F_j \right).$$

If two surfaces M_i, M_j are closed then by use of the Gauss-Bonnet formula

$$(8) \quad \tilde{R}_\Sigma = 2\pi\chi(\Sigma),$$

where $\chi(\cdot)$ is the Euler-Poincaré characteristic, we obtain the following type kinematic formula.

Theorem 1. *Let M_k ($k = i, j$) be two closed C^2 -smooth surfaces in $\mathbb{E}_\epsilon^3$. Denote by F_k, H_k the area and the mean curvature of M_k , respectively. Then we have*

$$(9) \quad \int_{IS_\epsilon^3} \left(\int_{M_i \cap gM_j} \kappa_\Gamma^2 ds \right) dg = 2\pi^3 \left\{ 3 \left(F_i \tilde{H}_j^2 + F_j \tilde{H}_i^2 \right) - 2\pi(F_i \chi(M_j) + F_j \chi(M_i)) + 2\epsilon F_i F_j \right\}.$$

§3. HADWIGER'S THEOREM IN SPACE $\mathbb{E}_\epsilon^3$

Let D_i and D_j be two connected domains in space $\mathbb{E}_\epsilon^3$, bounded by surfaces ∂D_i and ∂D_j , which we assume to be of class C^2 . In the case of the 3-sphere S^3 , ∂D_i and ∂D_j are assumed to be connected and simply connected. Moreover, we assume that D_i and D_j are such that, for all $g \in G$, the group of orientation preserving isometries of $\mathbb{E}_\epsilon^3$, the Euler-Poincaré characteristic $\chi(D_i \cap gD_j)$ of the intersection $D_i \cap gD_j$ has an upper bound N_0 , a finite integer. Let V_k ($k = i, j$), F_k , $\tilde{H}_k$ and $\tilde{H}_k^{(2)}$ be the volume of D_k , the surface area of D_k , the total mean curvature and the total square mean curvature of ∂D_k , respectively. We have

Theorem 2. *For the domains in a space $\mathbb{E}_\epsilon^3$ of nonnegative curvature ϵ (that is, space with constant curvature $\epsilon = 0, +1$), let $F_{\max} = \frac{1}{2} \max\{F_i, F_j\}$. In the case of S^3 , we assume that $F_k < 4\pi$ ($k = i, j$). Then a sufficient condition for D_i to contain D_j or for D_j to contain D_i is*

$$(10) \quad 8\pi(V_i\chi(D_j) + V_j\chi(D_i)) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) - \frac{N_0 \cdot 2\pi^2}{2\pi - \epsilon F_{\max}} \\ \times \{F_i F_j [3(\tilde{H}_i^2 F_j + \tilde{H}_j^2 F_i) - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) + 2\epsilon F_i F_j]\}^{\frac{1}{2}} > 0.$$

Moreover, if $V_j \leq V_i$, then D_j can be contained in D_i .

¹This formula, i.e., the analogue of C-S. Chen's formula, is due to R. Howard.

Proof. S. S. Chern's fundamental formula, Santaló's formula, respectively, read [1]

$$(11) \quad \int_{IS^3_\epsilon} \chi(D_i \cap gD_j) dg = 8\pi^2(V_i\chi(D_j) + V_j\chi(D_i)) + 2\pi(F_i\tilde{H}_j + F_j\tilde{H}_i),$$

$$(12) \quad \int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j)} L_{\Gamma_g} dg = 2\pi^3 F_i F_j,$$

where L_{Γ_g} is the volume of the intersection curve $\Gamma_g = \partial D_i \cap g\partial D_j$, i.e., the arc length of Γ_g . The curve Γ_g may be composed of several components (i.e., each of those is a simply closed curve).

For a disktype surface of area F bounded by a simple C^2 -smooth curve Γ , we have the following Fenchel-Teufel formula [11]:

$$(13) \quad \int_{\Gamma} |\kappa_{\Gamma}| ds \geq 2\pi - \epsilon F,$$

where κ_{Γ} is the geodesic curvature of Γ .²

By Hölder's inequality we have

$$(14) \quad \begin{aligned} 2\pi - \epsilon F_{\max} &\leq \int_{\Gamma_g} |\kappa_{\Gamma_g}| ds \leq \left(\int_{\Gamma_g} 1^2 \cdot ds \right)^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}} \\ &= (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}}, \end{aligned}$$

i.e.,

$$(15) \quad 2\pi - \epsilon F_{\max} \leq (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}}.$$

Integrating both sides of the above inequality with respect to kinematic density dg and using Hölder's inequality again we get

$$(16) \quad \begin{aligned} (2\pi - \epsilon F_{\max}) \int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j \neq \emptyset)} dg &\leq \int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j)} (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}} dg \\ &\leq \left(\int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j)} L_{\Gamma_g} dg \right)^{\frac{1}{2}} \left(\int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j)} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right) dg \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore by (9), (12) and (16) we have

$$(17) \quad \begin{aligned} \int_{IS^3_\epsilon(\partial D_i \cap g\partial D_j \neq \emptyset)} dg &\leq \frac{2\pi^3}{2\pi - \epsilon F_{\max}} \{F_i F_j [3(F_i \tilde{H}_j^2 + F_j \tilde{H}_i^2) \\ &\quad - 2\pi(F_i \chi(\partial D_j) + F_j \chi(\partial D_i)) + 2\epsilon F_i F_j]\}^{\frac{1}{2}}. \end{aligned}$$

From

$$(18) \quad \int_{IS^3_\epsilon(D_i \cap gD_j)} \chi(D_i \cap gD_j) dg \leq N_0 \int_{IS^3_\epsilon(D_i \cap gD_j \neq \emptyset)} dg,$$

²In the case of Euclidean space $\mathbb{R}^3$, the inequality is Fenchel's inequality. Inequality is due to Teufel when Γ is a curve in a 3-sphere S^3 .

we obtain the kinematic measure

$$\begin{aligned}
 & m\{g \in IS_\epsilon^3 : gD_j \subseteq D_i \text{ or } gD_j \supseteq D_i\} \\
 &= m\{g \in IS_\epsilon^3 : D_i \cap gD_j \neq \emptyset\} - m\{g \in IS_\epsilon^3 : \partial D_i \cap g\partial D_j \neq \emptyset\} \\
 (19) \quad &= \int_{IS_\epsilon^3(D_i \cap gD_j \neq \emptyset)} dg - \int_{IS_\epsilon^3(\partial D_i \cap g\partial D_j \neq \emptyset)} dg \\
 &\geq \frac{1}{N_0} [8\pi^2(V_i\chi(D_j) + V_j\chi(D_i)) + 2\pi(F_i\tilde{H}_j + F_j\tilde{H}_i)] - \frac{2\pi^3}{2\pi - \epsilon F_{\max}} \\
 &\quad \times \{F_iF_j[3(\tilde{H}_i^2F_j + \tilde{H}_j^2F_i) - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) + 2\epsilon F_iF_j]\}^{\frac{1}{2}}.
 \end{aligned}$$

If this lower bound is nonnegative then one domain can be moved to contain another. We complete the proof of Theorem 2. $\square$

Theorem 3. Let D_i, D_j be two domains in a hyperbolic space $\mathbb{H}^3$. Then a sufficient condition for D_i to contain D_j or for D_j to contain D_i is

$$\begin{aligned}
 (20) \quad & 8\pi(V_i\chi(D_j) + V_j\chi(D_i)) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) \\
 & - \pi N_0 \left\{ F_iF_j[3(F_i\tilde{H}_j^2 + F_j\tilde{H}_i^2) - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) - 2F_iF_j] \right\}^{\frac{1}{2}} > 0.
 \end{aligned}$$

Moreover, if $V_j \leq V_i$, then D_j can be contained in D_i .

Proof. For a disktype surface of area F bounded by a C^2 -smooth curve Γ in the 3-dimensional hyperbolic space $\mathbb{H}^3$, we have the following Brickell-Hsiung inequality [18]:

$$(21) \quad \int_{\Gamma} |\kappa_{\Gamma}| ds \geq 2\pi + F.$$

By Hölder's inequality and the weaker type of inequality (21) we have

$$\begin{aligned}
 (22) \quad & 2\pi \leq \int_{\Gamma_g} |\kappa_{\Gamma_g}| ds \leq \left(\int_{\Gamma_g} 1^2 \cdot ds \right)^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}} \\
 & \leq (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}},
 \end{aligned}$$

i.e.

$$(23) \quad 2\pi \leq (L_{\Gamma_g})^{\frac{1}{2}} \left(\int_{\Gamma_g} \kappa_{\Gamma_g}^2 ds \right)^{\frac{1}{2}}.$$

$\square$

Integrating both sides of inequality (23) over $IS_{-1}^3(\partial D_i \cap g\partial D_j \neq \emptyset)$ and using the analog of C-S. Chen's formula and Hölder inequality again we have

$$\begin{aligned}
 (24) \quad & \int_{IS_{-1}^3(\partial D_i \cap g\partial D_j \neq \emptyset)} dg \leq \pi^2 \{F_iF_j[3(F_i\tilde{H}_j^2 + F_j\tilde{H}_i^2) \\
 & - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) - 2F_iF_j]\}^{\frac{1}{2}}.
 \end{aligned}$$

By (18), (11) and (24) we have

$$\begin{aligned}
 (25) \quad & m\{g \in IS_{-1}^3 : gD_j \subseteq D_i \text{ or } gD_j \supseteq D_i\} \\
 &= m\{g \in IS_{-1}^3 : D_i \cap gD_j \neq \emptyset\} - m\{g \in IS_{-1}^3 : \partial D_i \cap g\partial D_j \neq \emptyset\} \\
 &= \int_{IS_{-1}^3(D_i \cap gD_j \neq \emptyset)} dg - \int_{IS_{-1}^3(\partial D_i \cap g\partial D_j \neq \emptyset)} dg \\
 &\geq \frac{1}{N_0} [8\pi^2(V_i\chi(D_j) + V_j\chi(D_i)) + 2\pi(F_i\tilde{H}_j + F_j\tilde{H}_i)] \\
 &\quad - \pi^2\{F_iF_j[3(\tilde{H}_i^2F_j + \tilde{H}_j^2F_i) - 2\pi(F_i\chi(\partial D_j) + F_j\chi(\partial D_i)) - 2F_iF_j]\}^{\frac{1}{2}}.
 \end{aligned}$$

Inequality (25) immediately leads to the proof of Theorem 3.

As in euclidean space we say that a subset S of a 3-sphere S^3 or a 3-hyperbolic space $\mathbb{H}^3$ is convex if $P, Q \in S$ implies that the geodesic segment joining P to Q also lies in S . A nonempty, compact, convex set of $\mathbb{E}_\epsilon^3$ is called a convex body. If D_i and D_j are convex bodies, we have $\chi(\partial D_i) = \chi(\partial D_j) = 2$ and $\chi(D_i) = \chi(D_j) = \chi(D_i \cap gD_j) = 1$ for almost all $g \in IS_\epsilon^3$. Then from Theorem 2 and Theorem 3 we immediately obtain the following

Theorem 4. *For convex bodies D_i and D_j in a space $\mathbb{E}_\epsilon^3$ of nonnegative constant curvature ϵ (i.e., $\epsilon = 0, +1$), let $F_{\max} = \frac{1}{2} \min\{F_i, F_j\}$. In the case of S^3 , we assume $F_k < 4\pi$ ($k = i, j$). Then a sufficient condition for D_i to contain, or to be contained in, D_j is*

$$\begin{aligned}
 (26) \quad & 8\pi(V_i + V_j) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) - \frac{2\pi^2}{2\pi - \epsilon F_{\max}} \\
 & \times \{F_iF_j[3(\tilde{H}_i^2F_j + \tilde{H}_j^2F_i) - 4\pi(F_i + F_j) + 2\epsilon F_iF_j]\}^{\frac{1}{2}} > 0.
 \end{aligned}$$

Moreover, if $V_i \leq V_j$, then D_i can be contained in D_j .

Theorem 5. *Let D_i and D_j be convex bodies in a hyperbolic space $\mathbb{H}^3$. Then a sufficient condition for D_i to contain, or to be contained in, D_j is*

$$\begin{aligned}
 (27) \quad & 8\pi(V_i + V_j) + 2(F_i\tilde{H}_j + F_j\tilde{H}_i) \\
 & - \pi\left\{F_iF_j[3(F_i\tilde{H}_j + F_j\tilde{H}_i) - 4\pi(F_i + F_j) - 2F_iF_j]\right\}^{\frac{1}{2}} > 0.
 \end{aligned}$$

Moreover, if $V_j \leq V_i$, then D_j can be contained in D_i .

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