THE MAXIMAL NORMAL p -SUBGROUP OF THE AUTOMORPHISM GROUP OF AN ABELIAN p -GROUP

JUTTA HAUSEN AND PHILLIP SCHULTZ

(Communicated by Ronald M. Solomon)

ABSTRACT. Let p be a prime number and let G be an abelian p -group. Let Δ be the maximal normal p -subgroup of $\text{Aut } G$ and ζ the maximal p -subgroup of its centre. Let $\mathfrak{t}$ be the torsion radical of $\mathcal{E}(G)$. Then $\Delta = (1 + \mathfrak{t})\zeta$. The result is new for $p = 2$ and 3 , and the proof is new and valid for all primes p .

§1. INTRODUCTION

Throughout, p is an arbitrary but fixed prime and G is an abelian p -group. We let $\mathcal{E} = \mathcal{E}(G)$ denote the endomorphism ring of G , $\text{Aut } G$ its automorphism group, and $\Delta = O_p(\text{Aut } G)$ the maximal normal p -subgroup of $\text{Aut } G$. The importance of Δ in determining the structure of $\text{Aut } G$, characterising the action of $\text{Aut } G$ on G and recovering invariants of G from $\text{Aut } G$ has long been recognized; see for example [3, Section 114], [6] and [1].

For $p \geq 5$, Δ considered as a group of units of $\mathcal{E}$ has been determined in a series of papers starting with Shoda [17] for finite groups, Freedman [2] for countable reduced groups, Hill [9] for totally projective groups and Hausen [5] for arbitrary p -groups: if $\mathfrak{t}$ denotes the torsion radical of $\mathcal{E}$ (i.e. $\mathfrak{t}$ is the ideal consisting of all elements of finite order in the Jacobson radical of $\mathcal{E}$), then $\Delta = 1 + \mathfrak{t}$.

Crucial for the proof is the following result, first stated in [5]. It was inspired by Freedman [2], stated by Leptin for $k = 1$ [11] and, in fact, is easily proven by induction on k following Leptin's arguments [11, p.101]. The Pierce radical of $\mathcal{E}$ is defined to be

$$\mathbf{P} = \{\varepsilon \in \mathcal{E} : p^n G[p]\varepsilon \leq p^{n+1}G \text{ for all integers } n \geq 0\}.$$

Clearly, $\mathbf{P}$ is a two-sided ideal in $\mathcal{E}$ containing the Jacobson radical of $\mathcal{E}$ [13, p. 289]. Let $1 + \mathbf{P}$ denote the coset containing 1 in the quotient ring $\mathcal{E}/\mathbf{P}$.

The Freedman–Leptin Lemma. *Let $p \geq 5$, let G be a reduced abelian p -group, and let $\alpha = 1 + \eta \in \text{Aut } G \cap (1 + \mathbf{P})$. Then, for any natural number k , $\alpha^{p^k} = 1$ if and only if $p^k \eta = 0$.*

For $p \geq 5$, the description of Δ proceeds as follows: the ideal $\mathfrak{t}$ is contained in the Jacobson radical of $\mathcal{E}$; since $\mathfrak{t}$ consists of all elements in $\mathbf{P}$ of finite order, it follows that $1 + \mathfrak{t}$ is a normal p -subgroup of $\text{Aut } G$ and thus is contained in Δ [3, Theorem 114.4]. Conversely, Δ induces the identity mapping in all Ulm factors

Received by the editors January 28, 1997.

1991 *Mathematics Subject Classification.* Primary 20K10, 20F28, 20K30.

$p^n G[p]/p^{n+1} G[p]$ where n is a non-negative integer [2], so that $\Delta \leq 1 + \mathbf{P}$, and the Freedman–Leptin Lemma proves $\Delta = 1 + \mathbf{t}$.

Two new phenomena make this approach impossible for the two smallest primes. First, the maximal p -subgroup of the center $\zeta \operatorname{Aut} G$ of $\operatorname{Aut} G$ is contained in Δ . While $[\zeta \operatorname{Aut} G]_p = 1$ if p is odd, $[\zeta \operatorname{Aut} G]_2 = \langle -1 \rangle$ if $p = 2$. But if G is an unbounded 2-group, multiplication by -2 is an endomorphism in P of infinite order, so -1 is not in $1 + \mathbf{t}$.

Second, while for primes $p \geq 5$ the set of all automorphisms in $1 + \mathbf{P}$ of finite order forms a group, in fact a p -group which must be normal in $\operatorname{Aut} G$ and hence equals $1 + \mathbf{t}$, we shall show by examples that this need not be the case when $p = 2$ or $p = 3$. For these primes, there exist p -groups G having an automorphism α of order p for which $\alpha - 1 \in \mathbf{P}$ has infinite additive order. This answers in the negative a problem posed by Leptin [11, p. 102]. Moreover, for $p = 2$, our example will be such that $\alpha \notin (1 + \mathbf{t}) \langle -1 \rangle$.

Nevertheless, we are able to prove the following theorem, which is valid for all primes:

Theorem. *Let p be a prime, let G be an abelian p -group, and let Δ be the maximal normal p -subgroup of the automorphism group of G . Then $\Delta = (1 + \mathbf{t})\zeta$, where ζ denotes the maximal p -subgroup of the centre of $\operatorname{Aut} G$. Thus, $\Delta = 1 + \mathbf{t}$ if $p \geq 3$ or G is bounded, and $\Delta = (1 + \mathbf{t}) \times \langle -1 \rangle$ if G is an unbounded 2-group.*

It is interesting to note that if G has an unbounded basic subgroup, then as an abstract ring, $\mathbf{t}$ determines G up to isomorphism; furthermore, every automorphism of $\mathbf{t}$ is induced by an automorphism of G [8]. One may well conjecture that the same is true of Δ . If true, this would be a far-reaching extension of the results of Leptin [10] and Liebert [12].

The strategy of the proof may be outlined as follows: it is easy to show that $1 + \mathbf{t} \leq \Delta$ (2.6). Of course, $\langle -1 \rangle \leq \Delta$ when $p = 2$. For the reverse inclusion, we first show that $\Delta \subseteq 1 + P$ and assume the existence of some $\alpha \in \Delta$ such that $\alpha - 1$ (and, for $p = 2$, $-\alpha - 1$, too) has infinite order. We then use an elaborate construction to produce mixed commutators of the form $[\gamma, \alpha]$ (and $[\beta, -\alpha]$ if $p = 2$) which have infinite order. Since they would have to belong to Δ , this is a contradiction.

Any unexplained notation can be found in the standard references [3] for abelian groups and [15] for general group theory. In particular, $[x, y] = x^{-1}y^{-1}xy$.

§2. PRELIMINARIES

We begin with the promised examples which show that for $p = 2$ and $p = 3$ no substitute for the Freedman–Leptin Lemma can be hoped for. The examples also illustrate a simple case of the strategy outlined above.

Examples 2.1. For each prime $p \in \{2, 3\}$, there exist an abelian p -group G and an automorphism $\alpha = 1 + \eta \in 1 + \mathbf{P}$ such that $\alpha^p = 1$ but η has infinite additive order. When $p = 2$, $\eta + 2$ also has infinite order. In addition, there exists $\beta \in \operatorname{Aut} G$ such that $[\beta, \alpha]$ has infinite multiplicative order.

Proof. If $p = 2$, let $G = \bigoplus_{i \in \mathbb{N}} (\langle a_i \rangle \oplus \langle b_i \rangle)$, where a_i has order 2^{i+1} and b_i has order 2^i . For each i , define $a_i \eta = b_i$ and $b_i \eta = -2b_i$. Then $\eta \in \mathbf{P}$, $2^k \eta \neq 0$ and $2^k(\eta + 2) \neq 0$ for all positive integers k . Furthermore, $\eta^2 + 2\eta = 0$.

If $p = 3$, let $G = \bigoplus_{i \in \mathbf{N}} (\langle a_i \rangle \oplus \langle b_i \rangle \oplus \langle c_i \rangle)$, where a_i, b_i and c_i have orders 3^{3i} , 3^{3i-1} and 3^{3i-2} , respectively. For each i , define $a_i \eta = b_i$, $b_i \eta = c_i$, and $c_i \eta = -3c_i - 3b_i$. Then $\eta \in \mathbf{P}$, $3^k \eta \neq 0$ for all k , and $\eta^3 + 3\eta^2 + 3\eta = 0$.

In either case, let $\alpha = 1 + \eta$. Then $\alpha^p = 1$, so $\alpha \in \text{Aut} G \cap (1 + \mathbf{P})$, and $\alpha \notin (1 + \mathbf{t})\zeta$.

We construct the automorphism β only for the case $p = 2$, the case $p = 3$ being similar. For each $i \geq 1$, define

$$a_i \varepsilon = 0 \text{ and } b_i \varepsilon = \begin{cases} a_{i-1} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Then $\varepsilon^2 = 0$, so $\beta = 1 + \varepsilon \in \text{Aut} G$ with inverse $1 - \varepsilon$. Since $\alpha^{-1} = \alpha$ and $\varepsilon \eta \varepsilon = 0$, a routine calculation shows that $[\beta, \alpha] = 1 + \psi$, where $\psi = \eta \varepsilon + \varepsilon \eta + \eta \varepsilon \eta$. But $\psi^2 = 0$, while ψ has infinite additive order. Hence $[\beta, \alpha]^k = 1 + k\psi$ for all $k \in \mathbf{N}$, so $[\beta, \alpha]$ has infinite multiplicative order. $\square$

We shall need some results of elementary number theory. The first one is well known.

Lemma 2.2. *Let r , n and k be positive integers such that n is not divisible by p and $p^r n \leq p^k$. Then*

- (1) *The maximal p -power dividing $\binom{p^k}{p^r n}$ is p^{k-r} .*
- (2) *If*

$$M = M(p, k, r, n) = \sum_{j=1}^{p^k} \binom{p^k}{j} (p^r n)^j,$$

then p^{k+r} divides M , and if p^{k+r+1} divides M , then $p = 2$ and $r = 1$.

Proof. The first assertion is easily proved in the spirit of [15, p. 39]. In order to verify the second, let j be an integer such that $1 \leq j \leq p^k$, and let $x_j = \binom{p^k}{j} p^{rj} n^j$. By (1), x_j is divisible by p^{k+rj} if $(j, p) = 1$. Suppose $j = p^t m$, with t and m positive integers and m relatively prime to p . By (1), p^{k-t+rj} divides x_j , and $k - t + rp^t m = k + r + r(p^t m - 1) - t \geq k + r$ since $r(p^t m - 1) \geq 2^t - 1 \geq t$. Thus, each x_j is divisible by p^{k+r} . Suppose that p^{k+r+1} divides M . Since p^{k+r+1} does not divide x_1 , there must exist $j \geq 2$ such that p^{k+r+1} does not divide x_j , and j cannot be relatively prime to p . Thus, $j = p^t m$ with t and m as above, and $k - t + rj = k + r$. It follows that

$$t = r(j - 1) = r(p^t m - 1) \geq r(2^t - 1) \geq 2^t - 1.$$

Since $t \geq 1$, this implies $t = 1 = r$ and $p = 2$. $\square$

The maximal normal p -subgroups of the automorphism groups of elementary and of divisible groups are known:

Lemma 2.3 ([2], [5]). *Let A be an elementary abelian and D a divisible p -group. Then $O_p(\text{Aut} A) = 1$, and $O_p(\text{Aut} D) = 1$ unless $p = 2$, in which case $O_2(\text{Aut} D) = \langle -1 \rangle$.*

Proof. The first assertion is Lemma 4.8 of [2]. Let D be divisible. By Theorem 6.5 of [5], every normal torsion subgroup of $\text{Aut} D$ is contained in the centre, which consists of the multiplications by p -adic units [3, 115.1]. It is well known that the

p -component of the group of p -adic units is trivial except when $p = 2$, in which case it is cyclic of order two generated by -1 [16, II.3.8]. $\square$

Corollary 2.4. *Let $G = H \oplus D$, where H is reduced and D is divisible. Let $\delta \in \Delta$. Then $\delta - 1 \in \mathbf{P}$, and $D(\delta - 1) = 0$ unless $p = 2$, in which case $\delta|_D \in \langle -1 \rangle$.*

Proof. Let n be a nonnegative integer. Every automorphism of $p^n G[p]/p^{n+1} G[p]$ is induced by an automorphism of G [3, 114.1]; clearly, every automorphism of D is induced by an automorphism of G . Thus, the result follows from Lemma 2.3. $\square$

From now on we shall assume that $G = H \oplus D$, where H is reduced and D is divisible.

The following subset of $\mathbf{P}$ will play an important rôle. It contains $\mathbf{t}$, and equals $\mathbf{t}$ if G is reduced. Let

$$\Phi = \{ \varepsilon \in \mathbf{P} \mid p^m G\varepsilon \leq D \text{ for some } m \in \mathbb{N} \text{ and } D\varepsilon = 0 \}.$$

Lemma 2.5. *Let $\varepsilon \in \Phi$ and let m be a natural number such that $p^m G\varepsilon \leq D$. Then $\varepsilon^{2m+3} = 0$, and $(1 + \varepsilon)^{p^n} = 1 + p^n \varepsilon$ for all integers $n \geq 3m + 2$.*

Proof. The hypotheses imply $G\varepsilon \leq G[p^m] + D$. Let $\overline{G} = G/D$ and let $\overline{\varepsilon}$ be the endomorphism of $\overline{G}$ induced by ε . Then $p^m \overline{\varepsilon} = 0$, so that

$$\overline{G}\overline{\varepsilon} \leq \overline{G}[p^m] \leq \overline{G}[p^{m+1}] = (G[p^{m+1}] + D)/D.$$

The restriction of $\overline{\varepsilon}$ to $\overline{G}[p^{m+1}]$ is in $\mathbf{P}(\overline{G}[p^{m+1}])$, which equals the Jacobson radical of $\mathcal{E}(\overline{G}[p^{m+1}])$ since $\overline{G}[p^{m+1}]$ is bounded [7]. By [14], $\mathbf{J}\mathcal{E}(\overline{G}[p^{m+1}])$ is nilpotent of class at most $2m + 1$. Thus, $(\overline{\varepsilon}|_{\overline{G}[p^{m+1}]})^{2m+1} = 0$, which implies $G[p^{m+1}]\varepsilon^{2m+1} \leq D$. Hence,

$$G\varepsilon^{2m+3} \leq (G[p^m] + D)\varepsilon^{2m+2} \leq D\varepsilon = 0.$$

Let n be an integer such that $n \geq 3m + 2$. Then

$$(1 + \varepsilon)^{p^n} = 1 + p^n \varepsilon + \sum_{j=2}^{p^n} \binom{p^n}{j} \varepsilon^j = 1 + p^n \varepsilon + \sum_{j=2}^{2m+2} \binom{p^n}{j} \varepsilon^j.$$

Fix $2 \leq j \leq 2m + 2$. By hypothesis, $p^m \varepsilon^2 = 0$. Let $j = p^r \cdot q$ for some integers r and q with p and q relatively prime. By Lemma 2.2, $\binom{p^n}{j} \varepsilon^j = p^{n-r} \ell \varepsilon^j$ for some integer ℓ not divisible by p . Suppose $n - r < m$. Then $r > n - m \geq 3m + 2 - m = 2m + 2$, which implies

$$j = p^r q \geq 2^r > r > 2m + 2.$$

Thus, $\binom{p^n}{j} \varepsilon^j = 0$. $\square$

While our Examples 2.1 demonstrate that the *only if* part of the Freedman-Leptin Lemma is lost in the general case, we do have the following weak version of the *if* part.

Proposition 2.6. *Let $\varepsilon \in \Phi$. The following conditions are equivalent:*

- (1) $(1 + \varepsilon)^{p^k} = 1$ for some positive integer k .
- (2) $\varepsilon \in \mathbf{t}$.

(3) $1 + \varepsilon \in \Delta$.

In particular, $1 + \mathfrak{t} \leq \Delta$.

Proof. Let $\varepsilon \in \Phi$ and let $m \in \mathbb{N}$ be such that $p^m G\varepsilon \leq D$. By 2.5, $(1 + \varepsilon)^{p^n} = 1 + p^n \varepsilon$ for all integers $n \geq 3m + 2$. Assume (1), and choose $n \geq \max(k, 3m + 2)$. Then $1 = 1 + p^n \varepsilon$, which implies (2). Conversely, assume (2). Then $p^\ell \varepsilon = 0$ for some natural number ℓ . Choose $n \geq \max(\ell, 3m + 2)$. Then $(1 + \varepsilon)^{p^n} = 1$. Thus, (1) and (2) are equivalent. Since $\mathfrak{t}$ is an ideal of $\mathcal{E}$ contained in the Jacobson radical of $\mathcal{E}$, it follows that $1 + \mathfrak{t}$ is a normal p -subgroup of $\text{Aut } G$. Hence, $1 + \mathfrak{t} \leq \Delta$, and (2) implies (3). Trivially, (3) implies (1). $\square$

Corollary 2.7. For every prime p and every p -group G , $(1 + \mathfrak{t})\zeta \leq \Delta$.

§3. THE REDUCED CASE

The following result will be crucial for the proof of our theorem. It shows that the construction of Examples 2.1 can be extended to deal with arbitrary p -groups.

Proposition 3.1. Let G be a reduced abelian p -group and let $\alpha = 1 + \eta \in \Delta$. Then $\eta \in \mathfrak{t}$, or $p = 2$ and $2 + \eta \in \mathfrak{t}$.

The proof will be by contradiction using an elaborate construction. Throughout, we shall assume the following

Situation and Notations: The p -group G is reduced, $\alpha = 1 + \eta$ is an element of Δ , and $k \in \mathbb{N}$ is a fixed integer such that $\alpha^{p^k} = 1$. Then

$$(3.2) \quad \sum_{j=1}^{p^k} \binom{p^k}{j} \eta^j = 0.$$

We let $\sigma = \alpha^{-1} - 1$. By Corollary 2.4

$$(3.3) \quad \eta, \sigma \in \mathbf{P}$$

and

$$(3.4) \quad \eta + \sigma + \eta\sigma = 0, \quad \eta\sigma = \sigma\eta.$$

Also, we let $\alpha' = -\alpha$ and put $\eta' = \alpha' - 1$, and $\sigma' = (\alpha')^{-1} - 1$. Then

$$(3.5) \quad \eta' = -(2 + \eta), \quad \sigma' = -(2 + \sigma).$$

The primed elements will come into play only when $p = 2$. Since the entire construction would have to be repeated, we consider them in any case.

We also fix a basic subgroup B of G and a canonical set S of (nonzero) independent generators. Thus $B = \bigoplus_{s \in S} \langle s \rangle = \bigoplus_{i \in \mathbb{N}} B_i$, where B_i is generated by the elements in S of order p^i . For each $n \in \mathbb{N}$, put $B^{(n)} = \bigoplus_{i=1}^n B_i$ and $G_n = p^n G + \bigoplus_{i \geq n+1} B_i$. Then, for each n , $G = B^{(n)} \oplus G_n$ [13, p. 280]. Choose a family $\{\pi_s\}_{s \in S}$ of canonical projections such that $t\pi_s = \delta_{st}s$, $t \in S$, and $G_n \pi_s = 0$ if $s \in B^{(n)}$. Define $G_0 = G$.

It will be convenient to write $e(x) = n$ if $x \in G$ and $o(x) = p^n$.

Lemma 3.6. Assume the situation and notations above and, furthermore, assume that both η and η' have infinite order. Then there exist natural numbers u'_i, u_i, v_i and elements a'_i, a_i, b_i in G belonging to the canonical set of generators of B such that, for each $i \in \mathbb{N}$, the following hold:

$$(1) \quad e(a'_i) = u'_i, \quad e(a_i) = u_i, \quad e(b_i) = v_i,$$

- (2) $e(a'_i\eta') \leq u'_i < 3u'_i < e(a_i\eta) \leq u_i < 3u_i < v_i$,
- (3) $v_{i-1} < e(a'_i\eta')$ (put $v_0 = 0$),
- (4) if $\pi_i : G \rightarrow \langle b_i \rangle$ denotes the canonical projection to $\langle b_i \rangle$ and r_i, m_i, r'_i, m'_i are integers such that $b_i\sigma\pi_i = p^{r_i}m_ib_i$, $b_i\sigma'\pi_i = p^{r'_i}m'_ib_i$, and m_i, m'_i are not divisible by p , then $r_i \notin [u_i - 2u_{i-1} - k, u_i - 2u_{i-1} - 1]$ and $r'_i \notin [u'_i - 2u'_{i-1} - k, u'_i - 2u'_{i-1} - 1]$ (put $u_0 = u'_0 = 0$).

Proof. We claim that, for each natural number n , there exist u'_i, u_i, v_i and canonical generators a'_i, a_i, b_i of B , $i = 1, \dots, n$, satisfying (1) through (4). The proof is by induction on n . Since the proof for $n = 1$ is almost identical to the inductive step from n to $n + 1$, it is left to the reader. Assume the claim holds for n . Since $p^{v_n}G\eta' \neq 0$, there exists i such that $p^{v_n}B_i\eta' \neq 0$, and there is a canonical generator x of B_i with $v_n < e(x\eta')$. Now $p^{3i+k}G\eta' \neq 0$; thus there exist j and a canonical generator y of B_j such that $3i + k < e(y\eta')$. Proceeding this way, there exist ℓ and m and canonical generators z and w of B_ℓ and B_m , respectively, such that

$$\begin{aligned} v_n < e(x\eta') \leq e(x) = i < 3i + k < e(y\eta') \leq e(y) = j \\ < 3j < e(z\eta) < 3\ell + k < e(w\eta) \leq e(w) = m. \end{aligned}$$

Since $p^{3m}G \neq 0$, it is possible to choose $v_{n+1} > 3m$ such that $B_{v_{n+1}} \neq 0$. Let b_{n+1} be a canonical generator of $B_{v_{n+1}}$. Then $e(b_{n+1}) = v_{n+1}$. Let π_{n+1} denote the canonical projection from G onto $\langle b_{n+1} \rangle$. Let $r'_{n+1}, r_{n+1}, m'_{n+1}, m_{n+1}$ be integers such that

$$b_{n+1}\sigma'\pi_{n+1} = p^{r'_{n+1}}m'_{n+1}b_{n+1} \quad \text{and} \quad b_{n+1}\sigma\pi_{n+1} = p^{r_{n+1}}m_{n+1}b_{n+1},$$

where m'_{n+1}, m_{n+1} are not divisible by p . If $r'_{n+1} \notin [i - 2u'_n - k, i - 2u'_n - 1]$, let $a'_{n+1} = x$. By construction, if $r'_{n+1} \in [i - 2u'_n - k, i - 2u'_n - 1]$, then $r'_{n+1} \notin [j - 2u'_n - k, j - 2u'_n - 1]$. In this case, let $a'_{n+1} = y$. Similarly, r_{n+1} cannot belong to both the intervals $[\ell - 2u_n - k, \ell - 2u_n - 1]$ and $[m - 2u_n - k, m - 2u_n - 1]$. Choose a_{n+1} to be either z or w depending on which is the case. One verifies that the augmented sets of integers and elements in G satisfy the requirements of the lemma. $\square$

We shall define two endomorphisms ε and ε' as follows: for each i , define $b_i\varepsilon_i = a_i$ and let $\varepsilon_i = \pi_i\varepsilon_i$. Since $G_{v_i}\pi_i = 0$, each ε_i is an endomorphism of G . Let $\varepsilon = \sum_{i \in \mathbb{N}} p^{u_{i-1}}\varepsilon_i$. By [13, 6.3], $\varepsilon \in \mathcal{E}$. Similarly, define $b_i\varepsilon'_i = a'_i$, $\varepsilon'_i = \pi_i\varepsilon'_i$, and let $\varepsilon' = \sum_{i \in \mathbb{N}} p^{u'_{i-1}}\varepsilon'_i \in \mathcal{E}$.

Lemma 3.7. *Assume the hypotheses of 3.6. Then, for every integer $i \geq 0$, $G_{v_i}[p^{u_{i+1}}]\varepsilon = 0 = G_{v_i}[p^{u_{i+1}}]\varepsilon'$ (Reminder: $v_0 = 0$ and $G_0 = G$).*

Proof. Let $i \geq 0$ be an integer and let $x \in G_{v_i}[p^{u_{i+1}}]$. Then $x\pi_j = 0$ for $j \leq i$. Let $j \geq i + 1$. Then $x\pi_j = p^{v_j - u_{i+1}}n_jb_j$ for some integer n_j . Thus, $x\pi_j\varepsilon = p^{v_j - u_{i+1} + u_{j-1}}n_ja_j$ which is zero since $v_j - u_{i+1} > 3u_j - u_{i+1} \geq 2u_j$. Thus, $x\varepsilon = \sum_{j \in \mathbb{N}} p^{u_{j-1}}x\pi_j\varepsilon_j = 0$. A similar argument holds for ε' . $\square$

Corollary 3.8. *For every $\phi \in \mathcal{E}$, $\varepsilon\phi\varepsilon = \varepsilon'\phi\varepsilon' = 0$. In particular, $\varepsilon^2 = 0 = (\varepsilon')^2$.*

Proof. Since G is reduced, it suffices to show the maps annihilate B , which will be the case if they annihilate every b_i . Let $\phi \in \mathcal{E}$. Note that $b_i\varepsilon\phi = p^{u_{i-1}}a_i\phi \in p^{u_{i-1}}B^{(v_{i-1})} \oplus G_{v_{i-1}}[p^{u_i}]$. The definition of ε together with Lemma 3.7 implies $b_i\varepsilon\phi\varepsilon = 0$. The identical argument shows that $b_i\varepsilon'\phi\varepsilon' = 0$. $\square$

We are ready for the

Proof of Proposition 3.1. Assume the situation and notations above. Also, assume by way of contradiction that the conclusion of 3.1 is false. Then η has infinite additive order, and when $p = 2$, so does η' . It follows that G is unbounded. Hence, no matter what prime p was given, both η and η' have infinite order, which means the previous auxiliary results are available.

Using Corollary 3.8, the maps $\gamma = 1 + \varepsilon$ and $\gamma' = 1 + \varepsilon'$ are automorphisms of G . Define $\delta = \gamma^{-1}\alpha^{-1}\gamma\alpha$ and $\delta' = (\gamma')^{-1}(\alpha')^{-1}\gamma'\alpha'$. Then $\delta \in \Delta$ and, if $p = 2$, so is δ' . An easy calculation using (3.4) and 3.8 shows that

$$\delta = 1 + \psi \quad \text{with} \quad \psi = \varepsilon\eta + \sigma\varepsilon + \sigma\varepsilon\eta \quad \text{and} \quad \psi^2 = 0,$$

and similarly,

$$\delta' = 1 + \psi' \quad \text{with} \quad \psi' = \varepsilon'\eta' + \sigma'\varepsilon' + \sigma'\varepsilon'\eta' \quad \text{and} \quad (\psi')^2 = 0.$$

Thus, for every $n \in \mathbb{N}$, $\delta^n = 1 + n\psi$. Since $\delta \in \Delta$, $\delta^{p^\ell} = 1$ and hence $p^\ell\psi = 0$ for some $\ell \in \mathbb{N}$. Choosing ℓ sufficiently large, we also have $p^\ell\psi' = 0$ if $p = 2$. Fix i such that both $u_{i-1} > \ell$ and $u'_{i-1} > \ell$. Then $p^{u_{i-1}}\psi = 0$ and, provided $p = 2$, $p^{u'_{i-1}}\psi' = 0$. Hence

$$(3.9) \quad 0 = p^{u_{i-1}}b_i\psi = p^{u_{i-1}}b_i\varepsilon\eta + p^{u_{i-1}}b_i\sigma\varepsilon + p^{u_{i-1}}b_i\sigma\varepsilon\eta.$$

Since $b_i\sigma \in B^{(v_i)} \oplus G_{v_i}[p^{v_i}]$, Lemma 3.7 implies that $b_i\sigma = \sum_{j=1}^i k_j b_j + x$, where the k_j are integers and $x\varepsilon = 0$. From (3.3) we conclude that p divides k_i and, by 3.6(4),

$$(3.10) \quad k_i = p^{r_i}m_i \quad \text{with} \quad r_i \geq 1, \quad (p, m_i) = 1.$$

It follows that $p^{u_{i-1}}b_i\sigma\varepsilon = p^{2u_{i-1}}k_ia_i$ and $p^{u_{i-1}}b_i\sigma\varepsilon\eta = p^{2u_{i-1}}k_ia_i\eta$. From (3.9) and $b_i\varepsilon\eta = p^{u_{i-1}}a_i\eta$ we deduce that $(1 + k_i)p^{2u_{i-1}}a_i\eta = -k_ip^{2u_{i-1}}a_i$. Let n be an integer such that $n(1 + k_i) \equiv -m_i \pmod{p^{u_i}}$. By (3.10), p does not divide n and $(p^{2u_{i-1}}a_i)\eta = p^{r_i}n \cdot p^{2u_{i-1}}a_i$, which by construction is nonzero. Hence, $r_i < u_i - 2u_{i-1}$. Also, for every positive integer j , $(p^{2u_{i-1}}a_i)\eta^j = (p^{r_i}n)^j \cdot p^{2u_{i-1}}a_i$. It follows from (3.2) that

$$(3.11) \quad p^{2u_{i-1}} \left[\sum_{j=1}^{p^k} \binom{p^k}{j} (p^{r_i}n)^j \right] a_i = p^{2u_{i-1}} M(p, k, r_i, n) a_i = 0.$$

Suppose either that $p \neq 2$, or that $p = 2$ and $r_i \geq 2$. Then, by Lemma 2.2, the highest p -power dividing $M(p, k, r_i, n)$ is p^{k+r_i} . Thus, 3.11 implies that $k + r_i \geq u_i - 2u_{i-1}$; above we saw that $r_i < u_i - 2u_{i-1}$. It follows that

$$r_i \in [u_i - 2u_{i-1} - k, u_i - 2u_{i-1} - 1],$$

contrary to our delicate construction 3.6(4). Consequently, we must have that $p = 2$ and $r_i = 1$. In this situation, the primed elements and maps have the identical properties. We have $(\delta')^{2^\ell} = 1$, $(1 + \psi')^{2^\ell} = 1 + 2^\ell\psi' = 1$, and the choice of i was such that

$$0 = 2^{u'_{i-1}}b_i\psi' = 2^{u'_{i-1}}b_i\varepsilon'\eta' + 2^{u'_{i-1}}b_i\sigma'\varepsilon' + 2^{u'_{i-1}}b_i\sigma'\varepsilon'\eta'.$$

Let the k'_j be integers such that $b_i\sigma' = \sum_{j=1}^i k'_j b_j + x'$ with $x'\varepsilon' = 0$. As before, letting n' be an integer such that $n'(1 + k'_i) \equiv -m'_i \pmod{2^{u'_i}}$, we have that n' is odd and, for every positive integer j , $2^{2u'_{i-1}}a'_i(\eta')^j = (2^{r'_i}n')^j \cdot 2^{2u'_{i-1}}a'_i$. By

construction, $r'_i < u'_i - 2u'_{i-1}$. As before, it follows that $2^{2u'_{i-1}}M(2, k, r'_i, n')a'_i = 0$. Observing (3.5) and the fact that $r_i = 1$, we conclude that

$$2^{r'_i}m'_ib_i = b_i\sigma'\pi_i = -b_i(2 + \sigma)\pi_i = -(2 + 2^{r_i}m_i)b_i = -2(1 + m_i)b_i.$$

But m_i is odd, hence $r'_i \geq 2$. It follows from Lemma 2.2 that $k + r'_i \geq u'_i - 2u'_{i-1}$. Thus, r'_i belongs to the interval $[u'_i - 2u'_{i-1} - k, u'_i - 2u'_{i-1} - 1]$, contrary to 3.6(4). This final contradiction concludes the proof. $\square$

Proposition 3.1 yields the most difficult part of our theorem:

Corollary 3.12. *If G is a reduced p -group, then $\Delta \leq (1 + \mathfrak{t})\zeta$.*

§4. PROOF OF THE THEOREM

As before, we let $G = H \oplus D$, where H is reduced and D is divisible. By Corollary 2.7, it suffices to show $\Delta \leq (1 + \mathfrak{t})\zeta$. By 3.12, this is the case when G is reduced. Thus, we may assume that $D \neq 0$. Let $\delta = 1 + \eta \in \Delta$. Since Δ induces a normal p -subgroup of $\text{Aut}(G/D)$, it follows from 3.12 that there exists a natural number m such that either $p^m G\eta \leq D$, or $p = 2$ and $2^m G(2 + \eta) \leq D$. If the latter is the case, replace δ by $-\delta = 1 - (2 + \eta)$. Thus, we may assume without loss of generality that $p^m G\eta \leq D$. By Corollary 2.4, $\eta \in \mathbf{P}$. If $D\eta = 0$, then $\eta \in \Phi$, and $\delta \in 1 + \mathfrak{t}$, by Proposition 2.6. Suppose $D\eta \neq 0$. It follows from Corollary 2.4 that $p = 2$ and $\delta|_D = -1$, the multiplication by -1 . Choose any $\theta \in \text{Hom}(H, D)$ and extend it to an endomorphism of G by defining $D\theta = 0$. Then $\theta^2 = 0$ and $\gamma = 1 + 2^m\theta \in \text{Aut } G$. Let $\alpha = \delta\gamma\delta^{-1}\gamma^{-1}$. Then $\alpha \in \Delta$ and $\alpha|_D = 1$. As before, let $\delta^{-1} = 1 + \sigma$. Then $\eta + \sigma + \eta\sigma = 0$. Let $x \in G$. Since $G\eta$ and $G\sigma$ both are contained in $G[2^m] + D$, on which γ acts as the identity, we have

$$x\alpha = x\delta\gamma\delta^{-1}\gamma^{-1} = x + 2^m x\theta\sigma + x\sigma + x\eta + x\eta\sigma,$$

which implies $x(\alpha - 1) = -2^{m+1}x\theta$. It follows that $\alpha - 1 \in \Phi$, and Proposition 2.6 implies that $\theta \in \mathfrak{t}$. We have shown that $\text{Hom}(H, D)$ is a torsion group, which, since $D \neq 0$, implies $2^n H = 0$ for some positive integer n . But then $2 + \eta \in \Phi$ and $-\delta = 1 - (2 + \eta) \in 1 + \mathfrak{t}$, completing the proof.

REFERENCES

1. R. P. Abraham, *Normal p -subgroups of the Automorphism Group of an Abelian p -group*, Journal of Algebra (to appear).
2. H. Freedman, *The automorphisms group of countable primary reduced abelian groups*, Proc. London Math. Soc. (3) **12** (1962), 77–99. MR **24**:A3215
3. L. Fuchs, *Infinite Abelian Groups*, Vol. 2, Academic Press, New York, 1973. MR **50**:2362
4. J. Hausen, *Near central automorphisms of abelian torsion groups*, Trans. Amer. Math. Soc. **174** (1972), 199–215. MR **46**:9193
5. J. Hausen, *On the normal structure of automorphism groups of abelian p -groups*, J. London Math. Soc. (2), **5** (1972), 409–413. MR **48**:2275
6. J. Hausen, *How automorphism groups reveal Ulm invariants*, J. Algebra **44** (1977), 9–28. MR **56**:499
7. J. Hausen and J. A. Johnson, *Characterization of the primary abelian groups, bounded modulo the divisible subgroup, by the radical of their endomorphism rings*, Archiv d. Math. **29** (1977), 566–570. MR **57**:6231
8. J. Hausen, C. E. Praeger and P. Schultz, *Most Abelian p -groups are determined by the Jacobson radical of their endomorphism rings*, Math. Z. **216** (1994), 431–436. MR **95d**:20094
9. P. Hill, *The automorphism groups of primary abelian groups*, Proc. London Math. Soc. (2), **22** (1971), 24–38. MR **43**:7508

10. H. Leptin, *Abelsche p -Gruppen und ihre Automorphismengruppen*, Math. Z., **73** (1960), 235–253. MR **22**:730
11. H. Leptin, *Einige Bemerkungen über die Automorphismen abelscher Gruppen*, in Proc. Colloq. Abelian Groups (Tihany, 1963), Budapest, 1964, pp. 99–104. MR **29**:5929
12. W. Liebert, *Isomorphic automorphism groups of primary Abelian groups*, Abelian group theory: Proceedings of the 1985 Oberwolfach Conference, Eds. R Göbel and E A Walker, Gordon and Breach, New York, 1987, pp. 9–31. MR **90g**:20085
13. R. S. Pierce, *Homomorphisms of primary abelian groups*, in Topics in Abelian Groups (J. M. Irwin and E. A. Walker, eds.), Scott, Foresman and Co., 1963, pp. 215–310. MR **31**:1299
14. C. E. Praeger and P. Schultz, *The Loewy length of the Jacobson radical of a bounded endomorphism ring*, in Abelian Groups and Non-commutative Rings, Amer. Math. Soc., Contemporary Mathematics 130, 1992, pp. 349–360. MR **93h**:16059
15. D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982. MR **84k**:20001
16. J. P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973. MR **49**:8956
17. K. Shoda, *Über die Automorphismen einer endlichen abelschen Gruppe*, Math. Ann. **100** (1928), 674–686.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204-3476
E-mail address: `hausen@uh.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS 6009, AUSTRALIA
E-mail address: `schultz@maths.uwa.edu.au`