A NOTE ON SEQUENCES LYING IN THE RANGE OF A VECTOR MEASURE VALUED IN THE BIDUAL

BEGOÑA MARCHENA AND CÁNDIDO PIÑEIRO

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let X be a Banach space. It is unknown if every subset A of X lying in the range of an X^{**} -valued measure is actually contained in the range of an X -valued measure. In this paper we solve this problem in the case when we consider only vector measures of bounded variation.

INTRODUCTION

Let X be a Banach space. It is still unknown if every subset A of X lying in the range of an X^{**} -valued measure is actually contained in the range of an X -valued measure. In this paper we only consider vector measures of bounded variation, and solve this problem by exhibiting a sequence (x_n) in the $\mathcal{L}_\infty$ Banach space Y of Bourgain and Delbaen [BD] that lies in the range of a Y^{**} -valued measure of bounded variation but is not contained in the range of a Y -valued measure with bounded variation. The proof is based in the following result that we prove in section 2:

A bounded sequence (x_n) in X lies inside the range of an X^{**} -valued measure with bounded variation iff $(\alpha_n x_n)$ is contained in the range of an X -valued measure of bounded variation for every $(\alpha_n) \in c_0$.

We start by explaining some basic notation used in this paper. In general, our operator and vector measure terminology and notation follow [Ps] and [DU]. We only consider real Banach spaces. If X is such a space, B_X will denote its closed unit ball. The phrase “range of an X -valued measure” always means a set of the form $rg(F) = \{F(A) : A \in \Sigma\}$, where Σ is a σ -algebra of subsets of a set Ω and $F : \Sigma \rightarrow X$ is countably additive.

THE MAIN STEP

For simplicity, we denote by $R_{bv}(X)$ the vector space of all sequences (x_n) in X lying in the range of an X -valued measure of bounded variation. We have obtained the following result.

Theorem 1. *Let (x_n) be a bounded sequence in X . The following statements are equivalent:*

- i) $(x_n) \in R_{bv}(X^{**})$,

Received by the editors October 14, 1996 and, in revised form, March 14, 1997.
 1991 *Mathematics Subject Classification.* Primary 46G10, 47B10.

- ii) $(\alpha_n x_n) \in R_{bv}(X)$ for every $(\alpha_n) \in c_0$,
- iii) $(\alpha_n x_n) \in R_{bv}(X^{**})$ for every $(\alpha_n) \in c_0$.

Proof. $i) \Rightarrow ii)$ Suppose (x_n) is a sequence lying in the range of a vector measure valued in the bidual space X^{**} and having bounded variation. By [Pi1, Lemma 2] the operator $\Sigma e_n^* \otimes x_n : \ell_1 \rightarrow X$ is integral (here (e_n^*) is the unit basis of ℓ_∞). If (α_n) is a null sequence of real numbers, the operator $\Sigma e_n^* \otimes \alpha_n x_n : \ell_1 \rightarrow X$ is the composition of $\Sigma e_n^* \otimes x_n$ with the diagonal compact operator $(\beta_n) \in \ell_1 \rightarrow (\alpha_n \beta_n) \in \ell_1$. So Grothendieck's Theorem [DU, p.252] tells us that $\Sigma e_n^* \otimes (\alpha_n x_n) : \ell_1 \rightarrow X$ is nuclear. Again it follows from [Pi1, Lemma 2] that $(\alpha_n x_n)$ belongs to $R_{bv}(X)$.

$ii) \Rightarrow iii)$ This is obvious.

$iii) \Rightarrow i)$ By hypotheses we can consider the linear map

$$U : (\alpha_n) \in c_0 \longrightarrow \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n \in I(\ell_1, X).$$

It is a standard argument to prove that U has closed graph. Since $\ell_1^* \simeq \ell_\infty$ has the metric approximation property, $\mathcal{N}(\ell_1, X)$ is isomorphically isometric to a subspace of $I(\ell_1, X)$ [J, p.410]. As U maps each finite sequence $(\alpha_1, \alpha_2, \dots, \alpha_n, 0, \dots)$ into a nuclear operator, it is easy to prove that the rank of U is contained in $\mathcal{N}(\ell_1, X)$. We also denote by U the operator

$$(\alpha_n) \in c_0 \longrightarrow \Sigma e_n^* \otimes \alpha_n x_n \in \mathcal{N}(\ell_1, X).$$

Its dual operator U^* takes $\mathcal{L}(X, \ell_1^{**})$ into ℓ_1 ; in particular, $\mathcal{K}(X, \ell_1)$ into ℓ_1 . Let us prove that $U^*(\Sigma x_n^* \otimes e_n) = (\langle x_n, x_n^* \rangle)_n$ for every $T = \Sigma x_n^* \otimes e_n \in \mathcal{K}(X, \ell_1)$ (here (e_n) is the unit basis of ℓ_1). Using the trace duality, we obtain the following equalities:

$$\begin{aligned} U^*(T)_m &= \langle u_m, U^*(T) \rangle = \langle U(u_m), T \rangle = \langle e_m^* \otimes x_m, T \rangle \\ &= \text{tr}(T \circ (e_m^* \otimes x_m)) = \langle (\langle x_n, x_n^* \rangle), e_m^* \rangle = \langle x_m, x_m^* \rangle \end{aligned}$$

for all $m \in \mathbb{N}$ and for all $T \in \mathcal{K}(X, \ell_1)$, (u_m) being the unit basis of c_0 . To conclude the proof it suffices to notice that the adjoint of the operator

$$U^* : \mathcal{K}(X, \ell_1) \longrightarrow \ell_1$$

is defined by

$$(\alpha_n) \in \ell_\infty \longrightarrow \Sigma e_n^* \otimes \alpha_n x_n \in I(\ell_1, X^{**}),$$

which can be proved in the same way using the trace duality. Then we have obtained that $\Sigma e_n^* \otimes x_n : \ell_1 \rightarrow X^{**}$ is integral and, therefore, so is $\Sigma e_n^* \otimes x_n : \ell_1 \rightarrow X$. $\square$

If we consider countable additive vector measures, there is not an analogous theorem. To see this, let X be a non-reflexive $\mathcal{L}_\infty$ space. By the non-reflexivity, there is a bounded sequence (x_n) in X that is not contained in the range of any X^{**} -valued measure. Nevertheless, by [PR, Theorem 3.6] $(\alpha_n x_n)$ lies inside the range of an X -valued measure for every $(\alpha_n) \in c_0$.

Using the same method as in the proof of Theorem 1, we are going to prove a partial result in this general context. As in [PR], we denote by $R(X)$ the vector space of all sequences $(x_n) \in X$ lying in the range of a vector measure. If (x_n) belongs to $R(X)$, we put $\|(x_n)\|_r = \inf \|m\|$, where the infimum is taken over all

vector measures m satisfying

$$\{x_n : n \in \mathbb{N}\} \subset rg(m).$$

$(R(X), \|\cdot\|_r)$ is itself a Banach space.

Proposition 2. *Suppose X is a Banach space satisfying the condition $\mathcal{K}(X, \ell_1) \subset \Pi_1(X, \ell_1)$, and (x_n) is a bounded sequence in X . The following statements are equivalent:*

- i) $(x_n) \in R(X)$,
- ii) $(\alpha_n x_n) \in R(X)$ for every $(\alpha_n) \in c_0$,
- iii) $(x_n) \in R_{bv}(X^{**})$.

Proof. First of all, notice that the inclusion $\Pi_1(X, \ell_1) \subset \mathcal{K}(X, \ell_1)$ holds for every Banach space X .

i) $\Rightarrow$ ii) This is obvious since the closed convex hull of a range is the range of another vector measure [DU, p.274].

ii) $\Rightarrow$ iii) In this part of the proof we follow the ideas of Theorem 1. We consider the linear map

$$U : (\alpha_n) \in c_0 \longrightarrow (\alpha_n x_n) \in R(X).$$

It is continuous because its graph is closed.

If $S = \sum x_n^* \otimes e_n \in \Pi_1(X, \ell_1)$, by [Pi2, Proposition 2] the linear form

$$\psi_s : (z_n) \in R(X) \longrightarrow \sum_{n=1}^{\infty} \langle z_n, x_n^* \rangle \in \mathbb{R}$$

is well-defined, continuous, and $\|\psi_s\| \leq \pi_1(S)$. Then the linear map

$$S = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in \Pi_1(X, \ell_1) \longrightarrow U^*(\psi_s) \in \ell_1$$

is continuous. In the same way as in Theorem 1 we can prove that this map is defined by

$$\sum x_n^* \otimes e_n \in \Pi_1(X, \ell_1) \longrightarrow (\langle x_n, x_n^* \rangle) \in \ell_1.$$

Since $K(X, \ell_1) = \Pi_1(X, \ell_1)$, dualizing again we have

$$(\alpha_n) \in \ell_{\infty} \longrightarrow \sum \alpha_n x_n \otimes e_n^* \in I(\ell_1, X^{**}).$$

This shows that $\sum x_n \otimes e_n^* : \ell_1 \rightarrow X$ is integral. So $(x_n) \in R_{bv}(X^{**})$.

iii) $\Rightarrow$ i) In [Pi1] it is proved that every sequence in X lying in the range of a vector measure of bounded variation and valued in some superspace of X actually belongs to $R(X)$. $\square$

In [MR] it is proved that the condition $K(X, Y) \subset \Pi_1(X, Y)$ implies that $\mathcal{L}(X, Y) = \Pi_1(X, Y)$, whenever X or Y has the bounded approximation property. So, we have the following equivalence:

$$K(X, \ell_1) \subset \Pi_1(X, \ell_1) \iff \mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1).$$

Recall that a Banach space X is called a G.T.-space if $\mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2)$ [Ps]. In [G, Theorem 2] it is proved that the following statements are equivalent:

- i) $\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1)$,
- ii) $\mathcal{L}(X^*, \ell_1) = \Pi_1(X^*, \ell_1)$.

From this equivalence the next result follows easily.

Proposition 3. *Let X be a Banach space. The following statements are equivalent:*

- i) $\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1)$,
- ii) X and X^* are $G.T.$ -spaces.

In fact, if a Banach space X satisfies one of the two above conditions, then we have

$$\mathcal{L}(X, \ell_1) = \Pi_1(X, \ell_1) = \Pi_2(X, \ell_1).$$

THE COUNTEREXAMPLE

Let Y be the $\mathcal{L}_\infty$ space of Bourgain and Delbaen [BD]. They proved that Y satisfies the Radon-Nikodým property and has a normalized unconditional basis (e_n) . We are going to show that (e_n) lies inside the range of a Y^{**} -valued measure of bounded variation, but it does not belong to $R_{bv}(Y)$. In view of Theorem 1, if we prove that $(\alpha_n e_n) \in R_{bv}(Y^{**})$ for every $(\alpha_n) \in c_0$, then the sequence (e_n) will belong to $R_{bv}(Y^{**})$. Notice that it suffices to consider null sequences (α_n) such that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$. Take such a sequence (α_n) . Since Y is an $\mathcal{L}_\infty$ -space, it follows from [PR] that $(\alpha_n e_n)$ lies in the range of a Y -valued measure. As $(\alpha_n e_n)$ is an unconditional basis of Y , by [AD] we have that $(\alpha_n e_n)$ is a weakly ℓ_2 -summable sequence. Then the map

$$A : (\beta_n) \in \ell_1 \longrightarrow \sum_{n=1}^{\infty} \beta_n \alpha_n e_n \in Y$$

is absolutely summing because it admits the factorization $A = B \circ I$, I being the inclusion map from ℓ_1 to ℓ_2 and B the operator defined by $B(\beta_n) = \sum_{n=1}^{\infty} \beta_n \alpha_n e_n$. Now recall that absolutely summing and integral operators into an $\mathcal{L}_\infty$ -space are the same [SR]. So, by [Pi1] the sequence $(\alpha_n e_n)$ lies in the range of a Y^{**} -valued measure having bounded variation. Then we have proved that (e_n) belongs to $R_{bv}(Y^{**})$. A final note in [Pi1] tells us that (e_n) is actually in the range of a Y -valued measure. Therefore, (e_n) is an unconditional basis of Y lying in the range of a vector measure. Again a call to [AD] tells us that (e_n) is a weakly ℓ_2 -summable sequence.

Finally, suppose (e_n) is contained in the range of a Y -valued measure with bounded variation. Then the operator $T : \ell_1 \longrightarrow Y$, defined by $T(\beta_n) = \sum \beta_n e_n \in Y$ for all $(\beta_n) \in \ell_1$, should be Pietsch-integral [Pi1]. Since Y is a Radon-Nikodým space, it follows from [DU, Theorem VII.4.8] that T is even nuclear. Nevertheless, T is not a compact operator.

REFERENCES

- [AD] R. Anantharaman and J. Diestel, *Sequences in the range of a vector measure*, Annales Societatis Mathematicae Polonae, Serie I: Comm. Math. **XXX** (1991), 221-235. MR **92g**:46049
- [BD] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ -spaces*, Acta Math. **145** (1980), 155-176. MR **82h**:46023
- [DU] J. Diestel and J.J. Uhl, Jr, *Vector measures*, Math. Surveys 15, Amer. Math. Soc, Providence, R.I, 1977. MR **56**:12216
- [G] J.C. García-Vázquez, *Tensor products of vector measures and sequences in the range of a vector measure*, Proc. Amer. Math. Soc. **124** (1996), 3459-3467. MR **97f**:46030
- [J] H. Jarchow, *Locally Convex Spaces*, B.G. Teubner, Stuttgart, 1981. MR **83h**:46008
- [LR] J. Lindenstrauss and H.P. Rosenthal, *The $\mathcal{L}_p$ spaces*, Israel J. Math. **7** (1969), 325-349. MR **42**:5012

- [MR] J.S. Morrel and J.R. Retherford, *P-Trivial Banach Spaces*, Studia Math. **XLIII**(1) (1972), 1-25. MR **47**:2321
- [Pi1] C. Piñeiro, *Sequences in the range of a vector measure with bounded variation*, Proc. Amer. Math. Soc. **123** (1995), 3329-3334. MR **96a**:46033
- [Pi2] C. Piñeiro, *On p -summable sequences in the range of a vector measure*, Proc. Amer. Math. Soc. **125** (1997), 2073-2082. MR **97i**:46082
- [PR] C. Piñeiro and L. Rodríguez-Piazza, *Banach spaces in which every compact lies inside the range of a measure*, Proc. Amer. Math. Soc. **114** (1992), 505-517. MR **92e**:46038
- [Ps] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, CBMS, vol.60, Amer. Math. Soc., Providence, R.I., 1986. MR **88a**:47020
- [SR] C.P. Steagall and J.R. Retherford, *Fully nuclear and completely nuclear operators with applications to $\mathcal{L}_1$ and $\mathcal{L}_\infty$ -spaces*, Transac. Amer. Math. Soc. **163** (1972), 457-491. MR **54**:3368

DEPARTAMENTO DE MATEMÁTICAS, ESCUELA POLITÉCNICA SUPERIOR, UNIVERSIDAD DE HUELVA, 21810 LA RÁBIDA, HUELVA, SPAIN
E-mail address: `candido@uhu.es`