EIGENVALUE PINCHING THEOREMS ON COMPACT SYMMETRIC SPACES

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ABSTRACT. We prove two first eigenvalue pinching theorems for Riemannian symmetric spaces (Theorems 1 and 2). As their application, we answer negatively a question raised by Elworthy and Rosenberg, who proposed to show that for every compact simple Lie group G with a bi-invariant Riemannian metric h on G with respect to $-\frac{1}{2}B$, B being the Killing form of the Lie algebra $\mathfrak{g}$, the first eigenvalue $\lambda_1(h)$ would satisfy

$$\sum_{j=1}^2 \sum_{\ell=3}^n |[v_j, v_\ell]|^2 > n(2\lambda_1(h) - 1),$$

for all orthonormal bases $\{v_j\}_{j=1}^n$ of tangent spaces of G (cf. Corollary 3). This problem arose in an attempt to give a spectral geometric proof that $\pi_2(G) = 0$ for a Lie group G .

§1. EIGENVALUE PINCHING THEOREMS

Lichnerowicz and Obata's celebrated theorem says (cf. [2]) that if the Ricci curvature of an n -dimensional compact Riemannian manifold (M, g) satisfies

$$\text{Ric}_g \geq n - 1,$$

then the first eigenvalue $\lambda_1(g)$ of the Laplacian Δ_g satisfies

$$\lambda_1(g) \geq n,$$

and equality holds if and only if (M, g) is the standard unit sphere (S^n, g_0) of curvature 1.

It is known (cf. [8], [7], [3], [1]) that the following eigenvalue pinching theorem holds:

Theorem. *Let (M, g) be a compact n -dimensional Riemannian manifold with sectional curvature $K_g \geq 1$. Then, there exists a constant $C(n) > 1$, depending only on n , such that, if $C(n)n \geq \lambda_1(g) \geq n$, M is homeomorphic to S^n .*

It would be a difficult problem to determine such a constant $C(n)$ explicitly. We would propose the conjecture that

$$C(n) \geq \left(\frac{4}{n} + 1\right) \frac{n-1}{n} > 1.$$

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Indeed, we prove under a strong symmetry condition,

Theorem 1. *Let (M, g) be a compact n -dimensional Riemannian manifold with Ricci curvature*

$$\text{Ric}_g \geq n - 1.$$

Assume that

$$\left(\frac{4}{n} + 1\right)(n - 1) \geq \lambda_1(g) \geq n,$$

and (M, g) is a simply connected, irreducible, Riemannian symmetric space. Then (M, g) is the standard sphere (S^n, g_0) .

Proof. All simply connected compact irreducible Riemannian symmetric spaces are divided into two classes: type I or type II (cf. [5]). The spaces of type II are of the form G/K with a compact simple Lie group G and some closed subgroup K , and the invariant metric is given by $g = \frac{1}{2(n-1)}h$, where h is a G -invariant Riemannian metric on $M = G/K$ induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G . Then the Ricci tensor Ric_h satisfies

$$\text{Ric}_h = \frac{1}{2}h.$$

All the first eigenvalues of (M, h) are listed in the tables in [9]. A space of type I is a compact simply connected simple Lie group G , and $g = \frac{1}{4(n-1)}g'$. Here g' is a bi-invariant metric on G induced from the Killing form of $\mathfrak{g}$ whose Ricci tensor satisfies

$$\text{Ric}_{g'} = \frac{1}{4}g',$$

and one can also find in [9] a list of the first eigenvalues of (G, g') . Due to these tables, one can see that all the first eigenvalues of (M, g) , except when (M, g) is the unit sphere (S^n, g_0) , are bigger than $\left(\frac{4}{n} + 1\right)(n - 1)$. $\square$

§2. ISOMETRIC IMMERSIONS

In [6], Lawson and Simons proved the following theorem.

Theorem. *Let (M, g) be an n -dimensional compact Riemannian manifold isometrically immersed in the unit sphere $S^N(1)$ with second fundamental form β . Assume that*

$$\sum_{j=1}^q \sum_{\ell=q+1}^n (2|\beta(v_j, v_\ell)|^2 - \langle \beta(v_j, v_j), \beta(v_\ell, v_\ell) \rangle) < q(n - q),$$

for every orthonormal base $\{v_j\}_{j=1}^n$ of $T_x M$ ($x \in M$). Then the homology groups of M vanish, $H_q(M, \mathbf{Z}) = H_{n-q}(M, \mathbf{Z}) = 0$.

In [4], Elworthy and Rosenberg proved

Theorem. *Let (M, g) be an n -dimensional compact Riemannian manifold isometrically minimally immersed in $S^N(1)$. Assume that*

$$\sum_{j=1}^q \sum_{\ell=q+1}^n K_g(v_j, v_\ell) > \frac{q(n - q)}{2},$$

for every orthonormal base $\{v_j\}_{j=1}^n$ of $T_x M$, $x \in M$. Then

$$H_q(M, \mathbf{Z}) = H_{n-q}(M, \mathbf{Z}) = 0.$$

In particular, if $K_g(v_j, v_\ell) > \frac{1}{2}$ for all $v_j, v_\ell \in T_x M$ ($x \in M$), $j < \ell$, then M is a homotopy sphere. Here $K_g(X, Y)$, for an orthonormal pair $X, Y \in T_x M$, means the sectional curvature of (M, g) for a plane in $T_x M$ defined by X and Y .

We show:

Theorem 2. Let (M, h) be a compact simply connected irreducible Riemannian symmetric space and h an invariant Riemannian metric induced from the Killing form of the Lie algebra of the isometry group of (M, h) . Assume that

$$(\#) \quad \sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) > \frac{n}{4}(2\lambda_1(h) - 1),$$

for all orthonormal bases $\{v_j\}_{j=1}^n$ of $T_x M$, $x \in M$. Then (M, h) is the unit sphere $(S^n, 2(n-1)g_0)$ of constant curvature $\frac{1}{2(n-1)}$.

Proof. In general, for a Riemannian manifold (M, g) ,

$$\begin{aligned} \sum_{j=1}^2 \sum_{\ell=3}^n K_g(v_j, v_\ell) &= \sum_{j=1}^2 \sum_{\ell=1}^n K_g(v_j, v_\ell) - \sum_{j,\ell=1}^2 K_g(v_j, v_\ell) \\ &= \text{Ric}_g(v_1) + \text{Ric}_g(v_2) - 2K_g(v_1, v_2) \\ &\geq 2(\inf \text{Ric}_g - \max K_g). \end{aligned}$$

For our case, since (M, h) satisfies $\text{Ric}_h = \frac{1}{2}h$,

$$\sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) = 1 - 2K_h(v_1, v_2),$$

which yields

$$0 \leq \inf \sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) = 1 - 2 \max K_h \leq 1.$$

Thus, the inequality $(\#)$ implies $1 \geq \frac{n}{4}(2\lambda_1(h) - 1)$.

But, if we put $g = \frac{1}{2(n-1)}h$, then $\text{Ric}_g = (n-1)g$, and the first eigenvalue $\lambda_1(g)$ of g satisfies

$$\left(\frac{4}{n} + 1\right)(n-1) \geq \lambda_1(g).$$

Therefore, we have Theorem 2, because of Theorem 1. $\square$

Theorem 2 immediately yields

Corollary 3. Let G be a compact simply connected simple Lie group, and h a bi-invariant Riemannian metric G induced from the Killing form B of the Lie algebra $\mathfrak{g}$ of G by identifying $G \cong G \times G / \Delta G$ as a symmetric space. That is,

$$h_x(X_x, Y_x) = -\frac{1}{2}B(X, Y), \quad X, Y \in \mathfrak{g},$$

where $X_x, Y_x \in T_x G$, $x \in G$. Assume that

$$\sum_{j=1}^2 \sum_{\ell=3}^n |[v_j, v_\ell]|^2 > n(2\lambda_1(h) - 1),$$

for every orthonormal base $\{v_j\}_{j=1}^n$ of $\mathfrak{g}$ with respect to $-\frac{1}{2}B$. Then (M, h) is $(SU(2), h)$.

Remark 4. (1) In [4, pp. 74-75], Elworthy and Rosenberg proposed to give a spectral geometric proof that $\pi_2(G) = 0$ for a Lie group G . The problem has been reduced to a question as to whether the above inequality in Corollary 3 would hold for all compact simply connected simple Lie group. But our Corollary 3 answers their question negatively.

(2) Let (M, h) be a compact simply connected irreducible Riemannian symmetric space. Let

$$\Phi_1 : (M, \frac{\lambda_1}{n}h) \rightarrow S^N(1)$$

be an isometric minimal immersion into the unit sphere defined by the first eigenfunctions of (M, h) . Let β be the second fundamental form and $\tilde{h} = \frac{\lambda_1}{n}h$. The inequality (#) in Theorem 2 is equivalent to

$$\sum_{j=1}^2 \sum_{\ell=3}^n K_{\tilde{h}}(v'_j, v'_\ell) > \frac{|\beta|^2}{2} + \frac{n}{2},$$

for every orthonormal base $\{v'_j\}_{j=1}^n$ of $T_x M$ ($x \in M$) relative to $\tilde{h}$.

(3) Note that

$$\max K_h = \max\{|\alpha|^2; \alpha \in \Sigma^+(G, K)\},$$

where $\Sigma^+(G, K)$ is the set of all positive roots α of G relative to a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ whose restrictions $\alpha|_{\mathfrak{a}}$ to $\mathfrak{a}$ do not vanish. Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is the Cartan decomposition of $\mathfrak{g}$, and the Cartan subalgebra $\mathfrak{t}$ is by definition a maximal abelian subalgebra of $\mathfrak{g}$ containing a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{m}$.

(4) If a Riemannian symmetric space (M, g) satisfies the inequality (#), then there might exist a q -th moment stable equation for $q \geq \frac{\dim M}{2}$ (cf. [4, p. 59]).

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