VECTOR BUNDLES WITH HOLOMORPHIC CONNECTION OVER A PROJECTIVE MANIFOLD WITH TANGENT BUNDLE OF NONNEGATIVE DEGREE

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ABSTRACT. For a projective manifold whose tangent bundle is of nonnegative degree, a vector bundle on it with a holomorphic connection actually admits a compatible flat holomorphic connection, if the manifold satisfies certain conditions. The conditions in question are on the Harder-Narasimhan filtration of the tangent bundle, and on the Neron-Severi group.

1. INTRODUCTION

Let E be a holomorphic vector bundle over a connected complex projective manifold M . Assume that E admits a holomorphic connection. Then a natural question to ask is whether E admits a flat holomorphic connection. Since all the rational Chern classes (of degree at least one) of a holomorphic vector bundle with a holomorphic connection vanish, there is no topological obstruction for the existence of a flat connection.

In this paper we consider this question for M satisfying the condition that the degree of the tangent bundle T_M is nonnegative with respect to some polarization on M .

Let

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_l \subset V_{l+1} = T_M$$

be the Harder-Narasimhan filtration of the tangent bundle T_M with respect to a polarization L on M .

In Theorem 2.4 we prove the following (degree of a coherent sheaf on M is computed using L):

Theorem A. *Assume that the degree of the tangent bundle $\deg T_M \geq 0$. Let E be a holomorphic vector bundle on M equipped with a holomorphic connection.*

- (1) *If $\deg(T_M/V_l) \geq 0$ then the holomorphic vector bundle E admits a compatible flat connection. (This inequality condition is satisfied if, for example, T_M is semistable, since $\deg T_M \geq 0$.)*

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- (2) Consider the case where T_M is not semistable. Assume that the maximal semistable subsheaf of T_M , namely V_1 , is locally free. If the rank of the Neron-Severi group, $NS(M)$, of M is 1, i.e.,

$$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q},$$

then E admits a compatible flat connection.

Under the assumptions either in part (1) or in part (2) of Theorem A, the vector bundle E turns out to be semistable with respect to L [Remark 2.12].

Generalizing the above question one may ask whether a holomorphic fiber bundle admitting a holomorphic connection actually admits a flat holomorphic connection. S. Murakami produced an example of a holomorphic fiber bundle over an abelian variety, with an abelian variety as fiber, such that the fiber bundle admits a holomorphic connection, but it does not admit any flat holomorphic connection [M1], [M2], [M3]. However part (1) of Theorem A implies that any holomorphic vector bundle over a projective manifold with trivial canonical line bundle, which admits a holomorphic connection, actually admits a flat holomorphic connection. Indeed, by a theorem of Yau [Ya] the tangent bundle of such a variety is semistable.

On the other hand, using a method of [Bi2], Theorem A can easily be generalized to principal G -bundles, where the structure group G is a connected affine algebraic reductive group over $\mathbb{C}$. The example of Murakami shows that it is essential for G to be noncompact.

2. CRITERIA FOR THE EXISTENCE OF A FLAT CONNECTION

Let $M/\mathbb{C}$ be a connected smooth projective variety of complex dimension d . We will denote by T_M (resp. Ω_M^1) the holomorphic tangent bundle (resp. cotangent bundle) of M .

For a holomorphic vector bundle V , the corresponding coherent analytic sheaf given by its local holomorphic sections will also be denoted by V . The basic facts about holomorphic structures used here can be found in [Ko].

A *holomorphic connection* on a holomorphic vector bundle E over M is a first order differential operator

$$(2.1) \quad D : E \longrightarrow \Omega_M^1 \otimes E$$

satisfying the following Leibniz condition:

$$(2.2) \quad D(fs) = fD(s) + df \otimes s$$

where f is a local holomorphic function on M and s is a local holomorphic section of E . Extend D as a first order operator

$$D : \Omega_M^{p,q} \otimes E \longrightarrow \Omega_M^{p+1,q} \otimes E$$

using the Leibniz rule. The *curvature* of D is defined to be

$$D^2 := D \circ D$$

which is a holomorphic section of $\Omega_M^2 \otimes \text{End } E$ ($\Omega_M^k := \bigwedge^k \Omega_M^1$). The notion of a holomorphic connection was introduced by M. Atiyah [At].

If $\bar{\partial}_E : E \longrightarrow \Omega_M^{0,1} \otimes E$ denotes the first order differential operator defining the holomorphic structure on E , then the operator

$$D + \bar{\partial}_E$$

is a connection on E in the usual sense. Moreover, the curvature of this connection is D^2 ; in particular, it is a holomorphic section of $\Omega_M^2 \otimes \text{End } E$. Conversely, the $(1, 0)$ part of a connection on E , such that the $(0, 1)$ part of it is $\bar{\partial}_E$ and its curvature is a holomorphic section of $\Omega_M^2 \otimes \text{End } E$, is actually a holomorphic connection.

In particular, if ∇ is a flat connection on a C^∞ complex vector bundle M , then the $(0, 1)$ part of the connection operator defines a holomorphic structure on E and the $(1, 0)$ part defines a holomorphic connection.

Let L be a polarization on M , or equivalently, L is an ample line bundle on M . For a coherent sheaf F on M , the *degree* of F , denoted by $\deg F$, is defined as follows ($d = \dim_{\mathbb{C}} M$):

$$\deg F := \int_M c_1(F) \cup c_1(L)^{d-1}.$$

A torsion-free coherent sheaf F is called *semistable* if for every (nonzero) coherent subsheaf $V \subset F$, the following inequality holds:

$$\frac{\text{rank } V}{\deg V} \leq \frac{\text{rank } F}{\deg F}.$$

Moreover, if the strict inequality holds for every proper coherent subsheaf V with F/V torsion-free, then F is called *stable*.

The quotient $\text{rank } F / \deg F$ is called the *slope* of F and is usually denoted by $\mu(F)$.

Any torsion-free coherent sheaf F admits a unique filtration by coherent subsheaves, known as the *Harder-Narasimhan filtration*, of the following type ([Ko], page 174, Theorem 7.15):

$$0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_k \subset F_{k+1} = F$$

where F_1 is the maximal semistable subsheaf of F . The Harder-Narasimhan filtration is determined by the property that F_{i+1}/F_i is the maximal semistable subsheaf of F/F_i . This implies that $\mu(F_{i+1}/F_i) < \mu(F_i/F_{i-1})$.

Let

$$(2.3) \quad 0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_l \subset V_{l+1} = T_M$$

be the Harder-Narasimhan filtration of the tangent bundle T_M .

A flat connection on a holomorphic vector bundle E on M is said to be *compatible* if the $(0, 1)$ part of the connection is $\bar{\partial}_E$ (equivalently, (local) flat sections are holomorphic sections). A compatible flat connection is same as a flat holomorphic connection.

Theorem 2.4. *Assume that the degree of the tangent bundle $\deg T_M \geq 0$. Let E be a holomorphic vector bundle on M equipped with a holomorphic connection.*

- (1) *If $\deg(T_M/V_l) \geq 0$ then the holomorphic vector bundle E admits a compatible flat connection. (This inequality condition is satisfied if, for example, T_M is semistable since $\deg T_M \geq 0$.)*
- (2) *Consider the case where T_M is not semistable. Assume that the maximal semistable subsheaf of T_M , namely V_1 , is locally free. If the rank of the Neron-Severi group, $NS(M)$, of M is 1, i.e.,*

$$H^{1,1}(M) \cap H^2(M, \mathbb{Q}) = \mathbb{Q},$$

then E admits a compatible flat connection.

Proof. Assume that $\deg(T_M/V_l) \geq 0$. Then from Lemma 2.1 of [Bi2] (also Remark 3.7(ii) of [Bi1]) we know that the vector bundle E is semistable. To be self-contained as much as possible we will quickly recall the proof of the semistability of E . Since E admits a holomorphic connection, Theorem 4 (page 192) of [At] says that all the (rational) Chern classes, $c_k(E)$, where $k \geq 1$, of E vanish. In particular $\deg E = 0$. Let W be the maximal semistable subsheaf of E . The key observation is that W is left invariant by the holomorphic connection operator D on E . Indeed, the homomorphism

$$(2.5) \quad W \longrightarrow \Omega_M^1 \otimes \frac{E}{W}$$

induced by D is $\mathcal{O}_M$ -linear (a simple consequence of the Leibniz identity (2.2)). The Harder-Narasimhan filtration of a tensor product is simply the tensor product of the corresponding Harder-Narasimhan filtrations. Applying this to $\Omega_M^1 \otimes (E/W)$, since the degree of any subsheaf of Ω_M^1 is nonpositive (this is equivalent to the assertion that the degree of a quotient sheaf of T_M is nonnegative, which, in turn, is warranted by the assumption that $\deg(T_M/V_l) \geq 0$), the slope of the maximal semistable subsheaf of $\Omega_M^1 \otimes (E/W)$ is less than or equal to $\mu(E/W)$. Finally from the general properties of Harder-Narasimhan filtrations we have $\mu(W) > \mu(E/W)$. If the image of the homomorphism in (2.5) is nonzero then the slope of the image is simultaneously at least $\mu(W)$ (recall that W is semistable) and as well as it is at most the slope of the maximal semistable subsheaf of $\Omega_M^1 \otimes (E/W)$. This contradicts the earlier observation that the slope of the maximal semistable subsheaf of $\Omega_M^1 \otimes (E/W)$ is strictly less than $\mu(W)$. Thus the homomorphism in (2.5) must be the zero homomorphism. In other words, W has an induced holomorphic connection. This implies that W is locally free of degree zero. So W cannot be a proper subsheaf of E . In other words, E must be semistable.

Since E is semistable with vanishing first and second Chern classes, the Corollary 3.10 (page 40) of [Si] implies that E admits a flat connection compatible with its holomorphic structure.

To prove part (2) of Theorem 2.4 we assume that T_M is not semistable. The maximal semistable subsheaf of T_M , namely V_1 (in (2.3)), is assumed to be locally free.

Our first step will be to prove that V_1 is closed under the Lie bracket operation on T_M . Towards this goal consider the homomorphism

$$(2.6) \quad \Gamma : V_1 \otimes V_1 \longrightarrow \frac{T}{V_1}$$

defined by composing the Lie bracket operation with the natural projection of T_M onto T_M/V_1 . Since the Lie bracket satisfies the Leibniz identity, namely

$$[fv, w] = f[v, w] - \langle df, w \rangle v,$$

where $\langle -, - \rangle$ denotes the obvious contraction, the map Γ is actually $\mathcal{O}_M$ -linear, i.e., Γ is a homomorphism of vector bundles.

Now we are given that $\mu(V_1) > \mu(T_M) \geq 0$. So

$$(2.7) \quad \mu(V_1 \otimes V_1) = 2\mu(V_1) > \mu(V_1) > \mu(V_2/V_1),$$

the last inequality being a general property of Harder-Narasimhan filtrations. The image of the homomorphism Γ is simultaneously a quotient of $V_1 \otimes V_1$ as well as a

subsheaf of T_M/V_1 . But V_2/V_1 , by definition, is the maximal semistable subsheaf of T_M/V_1 . So if $\Gamma \neq 0$ then

$$\mu(V_1 \otimes V_1) \leq \mu(\text{image } \Gamma) \leq \mu(V_2/V_1).$$

The first inequality is a consequence of the fact that $V_1 \otimes V_1$ is semistable. (A tensor product of semistable vector bundles is again semistable [MR], Remark 6.6 (iii).) This contradicts the inequality (2.7) unless $\text{image } \Gamma = 0$. But $\Gamma = 0$ is equivalent to V_1 being closed under the Lie bracket operation. In other words, V_1 is a nonsingular holomorphic foliation on M .

If E is semistable we may complete the proof of Theorem 2.4 by repeating the use of the Corollary 3.10 of [Si] as done in the proof of part (1) of Theorem 2.4. So we may, and we will, assume that E is not semistable. Let

$$0 = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_m \subset W_{m+1} = E$$

be the Harder-Narasimhan filtration of E .

Our next step will be to show that the sheaf W_1 has an induced *holomorphic partial connection* along the foliation V_1 . In other words, we want to show that the operator D in (2.1) induces an operator

$$(2.8) \quad D' : W_1 \longrightarrow V_1^* \otimes W_1$$

which satisfies the Leibniz condition (2.2); df in (2.2) is realized as a section of V_1^* in (2.8) by using the natural projection of Ω_M^1 onto V_1^* . The notion of a partial connection was introduced by R. Bott.

To construct D' first note that, by projecting Ω_M^1 onto V_1^* , the operator D in (2.1) induces an operator

$$(2.9) \quad D_1 : W_1 \longrightarrow V_1^* \otimes E.$$

Now projecting E onto E/W_1 , the operator D_1 in (2.9) induces an operator

$$D_2 : W_1 \longrightarrow V_1^* \otimes \frac{E}{W_1}.$$

The Leibniz identity (2.2) implies that D_2 is $\mathcal{O}_M$ -linear; i.e., the order of the differential operator D_2 is zero. In other words, D_2 is a homomorphism of vector bundles.

We will show that $D_2 = 0$ by following the steps of the argument for $\Gamma = 0$ (in (2.6)).

If $D_2 \neq 0$ then $\mu(\text{image}(D_2)) \geq \mu(W_1)$, since $\text{image}(D_2)$ is a quotient of the semistable sheaf W_1 . On the other hand, since

$$\text{image}(D_2) \subseteq V_1^* \otimes \frac{E}{W_1},$$

we conclude that the slope of $\text{image}(D_2)$ is at most the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

Thus if $D_2 \neq 0$, then $\mu(W_1)$ is less than or equal to the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$.

On the other hand, since V_1^* is semistable with strictly negative slope, the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than the slope of the maximal semistable subsheaf of E/W_1 – which in turn is strictly less than the slope of W_1 . Thus the slope of the maximal semistable subsheaf of $V_1^* \otimes (E/W_1)$ is strictly less than $\mu(W_1)$. This contradicts the inequality obtained in the previous paragraph. So we have $D_2 = 0$.

Since $D_2 = 0$, the differential operator D_1 in (2.9) induces a first order differential operator D' as in (2.8). Clearly D' satisfies the Leibniz identity, as D satisfies it.

The operator D' maps (local) holomorphic sections of W_1 to holomorphic sections of $V_1^* \otimes W_1$. So D' is a partial connection on W_1 along $V_1 \oplus T_M^{0,1}$ in the sense of [BB] (Sections 2 and 3); $T_M^{0,1}$ is the anti-holomorphic tangent bundle.

However, unfortunately, W_1 is not necessarily locally free. (A coherent sheaf equipped with a holomorphic connection must be locally free, but D' is only a partial connection.) To circumvent the problems caused by such a possibility of not being locally free, we will consider the determinant line bundle

$$d(W_1) := \det W_1 = \bigwedge^r W_1$$

where r is the rank of W_1 . The details of the construction of the determinant bundle of a torsion-free coherent sheaf can be found in Chapter 5, §6 of [Ko]. We note that the determinant bundle of a torsion-free sheaf is locally free of rank one, i.e., it is a line bundle.

The partial connection D' induces a partial connection on $d(W_1)$, which we will also denote by D' . More precisely, for a local section of $d(W_1)$

$$s := s_1 \wedge s_2 \wedge \dots \wedge s_r \in \Gamma(U, d(W_1))$$

the action of D' on it is defined as follows:

$$D'(s) := \sum_{j=1}^r s_1 \wedge \dots \wedge D'(s_j) \wedge \dots \wedge s_r.$$

It is straight-forward to check that the operator D' defined above satisfies the Leibniz identity. Thus D' is a partial holomorphic connection on $d(W_1)$ along V_1 .

We may extend the partial connection D' to an actual connection on $d(W_1)$ following [BB]. Fix a Kähler metric, say H , on M . Let ∇' be a hermitian connection on $d(W_1)$; the $(0,1)$ part of ∇' is assumed to be $\bar{\partial}_{d(W_1)}$. For any $v \in T_M^{1,0}$ let $v = v_1 \oplus v_2$ be the decomposition as $T_M^{1,0} = V_1 \oplus V_1^\perp$ using the metric H . For $v' \in T_M^{0,1}$ and a smooth section ϕ of $d(W_1)$ define:

$$\nabla_{v \oplus v'} \phi := \langle D' \phi, v_1 \rangle + \nabla'_{v_2} \phi + \langle \bar{\partial}_{d(W_1)} \phi, v' \rangle.$$

Clearly ∇ is a connection in the usual sense whose $(0,1)$ part coincides with $\bar{\partial}_{d(W_1)}$, and it is an extension of the partial connection D' .

Let

$$\mathcal{I} \subseteq \Omega_M^{1,1} \oplus \Omega_M^{0,2}$$

be the degree 2 component of the ideal, in the exterior algebra $\bigwedge (\Omega_M^{1,0} \oplus \Omega_M^{0,1})$, generated by the subspace of $\Omega_M^{0,1}$ that annihilates $\overline{V_1}$.

The following simple lemma will be useful:

Lemma 2.10. *The curvature ∇^2 , which is a smooth 2-form on M , is actually a section of $\mathcal{I} \oplus \Omega_M^{2,0}$.*

The proof of Lemma 2.10 is a simple computation. It is actually a straight-forward extension of (3.33), page 295 of [BB] to partial holomorphic connections (extension from partial flat connections). All we need to observe is that the cur-

vature of ∇' is of type $(1, 1)$ (since ∇' is assumed to be hermitian) and that the curvature of the partial connection D' is a holomorphic section of $\bigwedge^2 V_1^*$. Since the restriction of ∇ to a leaf of the foliation V_1 coincides with $D' + \bar{\partial}_{d(W_1)}$, the restriction of ∇^2 to a leaf is a section of $\bigwedge^2 V_1^*$. It is easy to see that this implies Lemma 2.10. $\square$

Continuing with the proof of Theorem 2.4, our next step will be to establish a lemma on vanishing of characteristic classes of $d(W_1)$, analogous to the Proposition (3.27), page 295, of [BB].

Lemma 2.11. *Let q be an integer with $q > \dim M - \dim V_1$. Then $c_1(d(W_1))^q = 0$.*

Proof of Lemma 2.11. The characteristic class $c_1(d(W_1))^q \in H^{q,q}(M)$, and it is represented by the differential form $(\nabla^2/2\pi\sqrt{-1})^q$. Since the space of forms on M admits Hodge decomposition, to prove Lemma 2.11 it is enough to show that the differential form $(\nabla^2)^q$ is a section of the vector bundle

$$\bigoplus_{j>q} \Omega_M^{j,2q-j}.$$

But Lemma 2.10 implies that $(\nabla^2)^q$ is indeed of the above type. To see this first note that by Lemma 2.10, both the $(1, 1)$ and the $(0, 2)$ part of ∇^2 is contained in the ideal generated by the subspace of $\Omega_M^{0,1}$ that annihilates $\overline{V_1}$. But the dimension of this annihilator is $\dim M - \dim V_1$. So the component of $(\nabla^2)^q$ in

$$\bigoplus_{j \leq q} \Omega_M^{j,2q-j}$$

vanishes identically. This completes the proof of the lemma. $\square$

To complete the proof of Theorem 2.4 we first note that the given condition that the rank of the Neron-Severi group, $NS(M)$, is 1 implies that if $(\omega)^j = 0$, where $\omega \in NS(M) \otimes_{\mathbb{Z}} \mathbb{Q} (= H^2(M, \mathbb{Q}) \cap H^{1,1}(M))$ and $1 \leq j \leq \dim_{\mathbb{C}} M$, then $\omega = 0$. This is simply because ω is a (possibly zero) rational multiple of the hyperplane class, and the j -th power of the hyperplane class is nonzero. Substituting $c_1(d(W_1))$ for ω and using Lemma 2.11 we get that $c_1(d(W_1)) = 0$. Thus we have

$$\deg W_1 = \deg d(W_1) = 0.$$

But W_1 is the maximal semistable subsheaf of E and $\deg E = 0$. This contradicts the assumptions that E is not semistable and that W_1 is the maximal semistable subsheaf of E . We already noted that if E is semistable then the Corollary 3.10 (page 40) of [Si] completes the proof of the theorem. This completes the proof of Theorem 2.4. $\square$

Remark 2.12. The proof of Theorem 2.4 shows that under the assumptions in either part 1 or part 2 of the statement of Theorem 2.4, the vector bundle E is actually semistable.

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