APPLICATIONS OF PSEUDO-MONOTONE OPERATORS WITH SOME KIND OF UPPER SEMICONTINUITY IN GENERALIZED QUASI-VARIATIONAL INEQUALITIES ON NON-COMPACT SETS

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ABSTRACT. Let E be a topological vector space and X be a non-empty subset of E . Let $S : X \rightarrow 2^X$ and $T : X \rightarrow 2^{E^*}$ be two maps. Then the generalized quasi-variational inequality (GQVI) problem is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. We shall use Chowdhury and Tan's 1996 generalized version of Ky Fan's minimax inequality as a tool to obtain some general theorems on solutions of the GQVI on a paracompact set X in a Hausdorff locally convex space where the set-valued operator T is either strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each non-empty finite subset A of X .

1. INTRODUCTION

If X is a set, we shall denote by 2^X the family of all non-empty subsets of X and by $\mathcal{F}(X)$ the family of all non-empty finite subsets of X . Let E be a topological vector space. We shall denote by E^* the continuous dual of E , by $\langle w, x \rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$ and by $\operatorname{Re}\langle w, x \rangle$ the real part of $\langle w, x \rangle$. If $X \subset E$, $S : X \rightarrow 2^X$ and $T : X \rightarrow 2^{E^*}$, the quasi-variational inequality problem (QVI) is to find a point $\hat{y} \in S(\hat{y})$ such that $\operatorname{Re}\langle T(\hat{y}), \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The QVI was first introduced by Bensoussan and Lions in 1973 (see, e.g., [2]) in connection with impulse control. Again, if we consider a set-valued map $T : X \rightarrow 2^{E^*}$, then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [4] in 1982 if $E = \mathbb{R}^n$ and by Shih and Tan [11] in 1985 if E is infinite dimensional.

In this paper, we shall use Chowdhury and Tan's generalized version [5, Theorem 2] of Ky Fan's minimax inequality [8, Theorem 1] as a tool to obtain some

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general theorems on solutions of the GQVI on a paracompact set X in a locally convex Hausdorff topological vector space where the set-valued operator T is strongly pseudo-monotone or pseudo-monotone and is upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$.

We shall use our following set-valued generalization of the classical pseudo-monotone operator. The classical definition of a pseudo-monotone operator was introduced by Brézis, Nirenberg and Stampacchia in [3]. For a slightly general definition of a pseudo-monotone operator we refer to [5, Definition 1].

Definition 1.1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. If $h : X \rightarrow \mathbb{R}$, then T is said to be (1) *h-pseudo-monotone* if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with $\limsup_\alpha [\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y \rangle + h(y_\alpha) - h(y)] \leq 0$, we have

$$\liminf_\alpha [\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - x \rangle + h(y_\alpha) - h(x)] \geq \inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)$$

for all $x \in X$; (2) *pseudo-monotone* if T is h -pseudo-monotone with $h \equiv 0$.

2. GENERALIZED QUASI-VARIATIONAL INEQUALITIES FOR STRONGLY PSEUDO-MONOTONE OPERATORS

In this section we shall introduce the notion of strongly pseudo-monotone operators and obtain some general theorems on solutions of the GQVI on paracompact sets in locally convex Hausdorff topological vector spaces.

We shall begin with the following:

Definition 2.1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \rightarrow 2^{E^*}$. If $h : X \rightarrow \mathbb{R}$, then T is said to be (1) *strongly h-pseudo-monotone* if for each continuous function $\theta : X \rightarrow [0, 1]$, for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y with

$$\limsup_\alpha [\theta(y_\alpha) (\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - y \rangle + h(y_\alpha) - h(y))] \leq 0$$

we have

$$\begin{aligned} \limsup_\alpha [\theta(y_\alpha) (\inf_{u \in T(y_\alpha)} \operatorname{Re}\langle u, y_\alpha - x \rangle + h(y_\alpha) - h(x))] \\ \geq [\theta(y) (\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x))] \end{aligned}$$

for all $x \in X$; (2) *strongly pseudo-monotone* if T is strongly h -pseudo-monotone with $h \equiv 0$.

Clearly, every strongly pseudo-monotone operator is also a pseudo-monotone operator as defined in [5].

Proposition 2.1. Let X be a non-empty subset of a topological vector space E . If $T : X \rightarrow E^*$ is monotone and continuous from the relative weak topology on X to the weak* topology on E^* , then T is strongly pseudo-monotone.

Proof. Let us consider any arbitrary continuous function $\theta : X \rightarrow [0, 1]$. Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X and $y \in X$ with $y_\alpha \rightarrow y$ (and

$$\limsup_\alpha [\theta(y_\alpha) (\operatorname{Re}\langle Ty_\alpha, y_\alpha - y \rangle)] \leq 0).$$

Then for any $x \in X$ and $\epsilon > 0$, there are $\beta_1, \beta_2 \in \Gamma$ with $|\theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - y \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_1$ and $|\theta(y_\alpha)Re\langle Ty_\alpha - Ty, y - x \rangle| < \frac{\epsilon}{2}$ for all $\alpha \geq \beta_2$. Choose $\beta_0 \in \Gamma$ with $\beta_0 \geq \beta_1, \beta_2$. Thus

$$\begin{aligned} \theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - x \rangle &= \theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - y \rangle + \theta(y_\alpha)Re\langle Ty_\alpha, y - x \rangle \\ &\geq \theta(y_\alpha)Re\langle Ty, y_\alpha - y \rangle + \theta(y_\alpha)Re\langle Ty_\alpha, y - x \rangle \\ &= \theta(y_\alpha)Re\langle Ty, y_\alpha - y \rangle + \theta(y_\alpha)Re\langle Ty_\alpha - Ty, y - x \rangle + \theta(y_\alpha)Re\langle Ty, y - x \rangle \\ &> -\frac{\epsilon}{2} - \frac{\epsilon}{2} + \theta(y_\alpha)Re\langle Ty, y - x \rangle \text{ for all } \alpha \geq \beta_0 \end{aligned}$$

so that $\inf_{\alpha \geq \beta_0} \theta(y_\alpha)Re\langle Ty_\alpha, y_\alpha - x \rangle \geq -\epsilon + \inf_{\alpha \geq \beta_0} \theta(y_\alpha)Re\langle Ty, y - x \rangle$. It follows that $\limsup_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x \rangle \geq \liminf_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x \rangle \geq -\epsilon + \theta(y)Re\langle Ty, y - x \rangle$. As $\epsilon > 0$ is arbitrary,

$$\limsup_\beta \theta(y_\beta)Re\langle Ty_\beta, y_\beta - x \rangle \geq \theta(y)Re\langle Ty, y - x \rangle.$$

Hence T is strongly pseudo-monotone. $\square$

We shall now establish the following result:

Theorem 2.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex subset of E and $h : E \rightarrow \mathbb{R}$ be convex. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be strongly h -pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} Re\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle w, y - x \rangle + h(y) - h(x)]$. Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set $V_p := \{y \in X : Re\langle p, y \rangle - \sup_{x \in S(y)} Re\langle p, x \rangle > 0\}$.

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 1 in [11] and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [7]); that is, for each $p \in E^*$, $\beta_p : X \rightarrow [0, 1]$ and $\beta_0 : X \rightarrow [0, 1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{\text{support } \beta_0, \text{support } \beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \sum_{p \in E^*} \beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in \mathcal{F}(X)$, h is

continuous on $co(A)$ (see e.g. [10, Corollary 10.1.1, p.83]). Define $\phi : X \times X \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)$ is lower semicontinuous on $co(A)$ by Lemma 3 in [5] and the fact that h is continuous on $co(A)$ and therefore the map $y \mapsto \beta_0(y) [\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)]$ is lower semicontinuous on $co(A)$ by Lemma 3 in [12]. Also for each fixed $x \in X$, $y \mapsto \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle$ is continuous on X . Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \phi(x, y) \leq 0$. Indeed, if this were false, then for some $A = \{x_1, \dots, x_n\} \in \mathcal{F}(X)$ and some $y \in co(A)$ (say $y = \sum_{i=1}^n \lambda_i x_i$ where $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$), we have $\min_{1 \leq i \leq n} \phi(x_i, y) > 0$. Then for each $i = 1, \dots, n$,

$$\beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re} \langle w, y - x_i \rangle + h(y) - h(x_i) \right] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x_i \rangle > 0$$

so that

$$\begin{aligned} 0 = \phi(y, y) &= \beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re} \langle w, y - \sum_{i=1}^n \lambda_i x_i \rangle + h(y) - h\left(\sum_{i=1}^n \lambda_i x_i\right) \right] \\ &\quad + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - \sum_{i=1}^n \lambda_i x_i \rangle \\ &\geq \sum_{i=1}^n \lambda_i \left(\beta_0(y) \left[\min_{w \in T(y)} \operatorname{Re} \langle w, y - x_i \rangle + h(y) - h(x_i) \right] \right. \\ &\quad \left. + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x_i \rangle \right) > 0, \end{aligned}$$

which is a contradiction.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

Then for $t = 0$ we have $\phi(y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha) \left[\min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] + \sum_{p \in E^*} \beta_p(y_\alpha) \operatorname{Re} \langle p, y_\alpha - y \rangle \leq 0$$

for all $\alpha \in \Gamma$. Hence

$$\begin{aligned} &\limsup_{\alpha} [\beta_0(y_\alpha) \left(\min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right)] \\ &\quad + \liminf_{\alpha} \left(\sum_{p \in E^*} \beta_p(y_\alpha) \operatorname{Re} \langle p, y_\alpha - y \rangle \right) \\ &\leq \limsup_{\alpha} [\beta_0(y_\alpha) \left(\min_{w \in T(y_\alpha)} \operatorname{Re} \langle w, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right) \\ &\quad + \sum_{p \in E^*} \beta_p(y_\alpha) \operatorname{Re} \langle p, y_\alpha - y \rangle] \leq 0. \end{aligned}$$

Therefore $\limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \leq 0$. Since T is strongly h -pseudo-monotone, we have

$$\begin{aligned} & [\limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \geq \beta_0(y)(\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)). \end{aligned}$$

Thus

$$\begin{aligned} (2.1) \quad & \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \\ & \geq \beta_0(y)(\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)) + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle. \end{aligned}$$

For $t = 1$ we have $\phi(x, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_{\alpha})[\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \quad + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \\ & \leq \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)) \\ & \quad + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} (2.2) \quad & \limsup_{\alpha} [\beta_0(y_{\alpha})(\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \quad + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \leq 0. \end{aligned}$$

Hence by (2.1) and (2.2), we have $\phi(x, y) \leq 0$.

(4) By hypothesis, there exist a non-empty compact (and therefore closed) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for each $y \in X \setminus K$. Thus for each $y \in X \setminus K$,

$$\beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)] > 0$$

whenever $\beta_0(y) > 0$ and $\operatorname{Re}\langle p, y - x_0 \rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently,

$$\begin{aligned} \phi(x_0, y) &= \beta_0(y)[\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0)] \\ & \quad + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x_0 \rangle > 0 \end{aligned}$$

for all $y \in X \setminus K$.

Then ϕ satisfies all hypotheses of Theorem 2 in [5]. Hence by Theorem 2 in [5], there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leq 0$ for all $x \in X$; i.e.,

$$(2.3) \quad \beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - x \rangle \leq 0$$

for all $x \in X$.

If $\gamma(\hat{y}) = 0$, choose any $\hat{x} \in S(\hat{y})$; if $\gamma(\hat{y}) > 0$, choose any $\hat{x} \in S(\hat{y})$ such that $\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \geq \frac{\gamma(\hat{y})}{2} > 0$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$; it follows that

$$\beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence $\text{Re}\langle p, \hat{y} \rangle > \sup_{x \in S(\hat{y})} \text{Re}\langle p, x \rangle \geq \text{Re}\langle p, \hat{x} \rangle$ so that $\text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0$. Then note that $\beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0$ whenever $\beta_p(\hat{y}) > 0$ for $p \in E^*$.

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x}, \hat{y}) = \beta_0(\hat{y}) \left[\inf_{w \in T(\hat{y})} \text{Re}\langle w, \hat{y} - \hat{x} \rangle + h(\hat{y}) - h(\hat{x}) \right] + \sum_{p \in E^*} \beta_p(\hat{y}) \text{Re}\langle p, \hat{y} - \hat{x} \rangle > 0,$$

which contradicts (2.3). This contradiction proves Step 1.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that $\text{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

Note that for each fixed $x \in S(\hat{y})$, $w \mapsto \text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, $x \mapsto \text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)$ is concave on $S(\hat{y})$. Thus by Kneser's Minimax Theorem in [9] (see also Aubin [1, pp.40-41]), we have

$$\begin{aligned} & \min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [\text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \\ &= \max_{x \in S(\hat{y})} \min_{w \in T(\hat{y})} [\text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)]. \end{aligned}$$

Hence $\min_{w \in T(\hat{y})} \max_{x \in S(\hat{y})} [\text{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0$ by Step 1. Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that $\text{Re}\langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$. $\square$

If X is compact, we obtain the following immediate consequence of Theorem 2.1:

Theorem 2.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \rightarrow \mathbb{R}$ be convex. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be strongly h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose the set $\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$ is open in X . Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Note that if X is also bounded in Theorem 2.1, the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X , then the set Σ in Theorem 2.1 is always open in X as can be seen in the proof of the following:

Theorem 2.3. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \rightarrow \mathbb{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be strongly h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each*

$$y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\},$$

T is upper semicontinuous at y from the relative topology on X to the strong topology on E^ . Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and*

$$\inf_{w \in T(y)} \text{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$$

for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. By virtue of Theorem 2.1, we need only show that the set

$$\Sigma := \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$$

is open in X . Indeed, let $y_0 \in \Sigma$; then there exists $x_0 \in S(y_0)$ such that $\alpha := \inf_{w \in T(y_0)} \text{Re}\langle w, y_0 - x_0 \rangle + h(y_0) - h(x_0) > 0$.

Let $W := \{w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2 \rangle| < \frac{\alpha}{6}\}$. Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(y_0) + W$ is an open neighborhood of $T(y_0)$ in E^* . Since T is upper semicontinuous at y_0 in X , there exists an open neighborhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Now, the rest of the proof is similar to the proof of Theorem 2.2 in [6]. Hence by the rest of the proof of Theorem 2.2 in [6], Σ is open in X . This proves the theorem. $\square$

If X is compact, we obtain the following immediate consequence of Theorem 2.3:

Theorem 2.4. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \rightarrow \mathbb{R}$ be convex. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be strongly h -pseudo-monotone and be upper semicontinuous from $\text{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \text{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\text{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

We remark here that in Theorems 2.1-2.4, the condition “ $h : E \rightarrow \mathbb{R}$ be convex” can be replaced by the condition “ $h : X \rightarrow \mathbb{R}$ be convex such that $h|_{\text{co}(A)}$ is continuous for each $A \in \mathcal{F}(X)$ ”.

3. GENERALIZED QUASI-VARIATIONAL INEQUALITIES FOR PSEUDO-MONOTONE OPERATORS

In this section we shall obtain some existence theorems of generalized quasi-variational inequalities for pseudo-monotone operators (Definition 1.1) on paracompact convex sets.

We shall first establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \rightarrow \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $co(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set $\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$ is open in X . Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. We divide the proof into two steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} \operatorname{Re}\langle w, \hat{y} - x \rangle + h(\hat{y}) - h(x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x) > 0$; that is, $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by Hahn-Banach separation theorem, there exists $p \in E^*$ such that $\operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0$. For each $y \in X$, set $\gamma(y) := \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)]$. Let $V_0 := \{y \in X : \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set $V_p := \{y \in X : \operatorname{Re}\langle p, y \rangle - \sup_{x \in S(y)} \operatorname{Re}\langle p, x \rangle > 0\}$.

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 1 in [11] and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X . Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [10, Corollary 10.1.1, p.83]). Define $\phi : X \times X \rightarrow \mathbb{R}$ by

$$\phi(x, y) = \beta_0(y) [\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle$$

for each $x, y \in X$. Then we have the following.

(1) The same argument in proving (1) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \mapsto \phi(x, y)$ is lower semicontinuous on $co(A)$.

(2) The same argument in proving (2) in the proof of Theorem 2.1 shows that for each $A \in \mathcal{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \phi(x, y) \leq 0$.

(3) Suppose $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in X converging to y with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

Case 1. $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \rightarrow 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha \in \Gamma}$ is a bounded net, it follows that

$$(3.1) \quad \limsup_{\alpha} [\beta_0(y_\alpha) (\min_{w \in T(y_\alpha)} \operatorname{Re}\langle w, y_\alpha - x \rangle + h(y_\alpha) - h(x))] = 0.$$

Also $\beta_0(y)[\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] = 0$. Thus

$$\begin{aligned}
 (3.2) \quad & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \\
 &= \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \quad (\text{by (3.1)}) \\
 &= \beta_0(y) [\min_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle.
 \end{aligned}$$

For $t = 1$ we have $\phi(x, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$(3.3) \quad \beta_0(y_{\alpha}) [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned}
 & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\
 & \quad + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \\
 & \leq \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)) \\
 & \quad + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - x \rangle] \\
 & \leq 0 \quad (\text{by (3.3)}).
 \end{aligned}$$

Thus

$$(3.4) \quad \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re}\langle p, y - x \rangle \leq 0.$$

Hence by (3.2) and (3.4), we have $\phi(x, y) \leq 0$.

Case 2. $\beta_0(y) > 0$.

Since $\beta_0(y_{\alpha}) \rightarrow \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_{\alpha}) > 0$ for all $\alpha \geq \lambda$. Then for $t = 0$ we have $\phi(y, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_{\alpha}) [\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - y \rangle \leq 0$$

for all $\alpha \in \Gamma$. Thus

$$(3.5) \quad \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re}\langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)) + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re}\langle p, y_{\alpha} - y \rangle] \leq 0.$$

Hence

$$\begin{aligned} & \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \\ & \quad + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - y \rangle] \\ & \leq \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \\ & \quad + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - y \rangle \\ & \leq 0 \quad (\text{by (3.5)}). \end{aligned}$$

Since $\liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - y \rangle] = 0$, we have

$$(3.6) \quad \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))] \leq 0.$$

Since $\beta_0(y_{\alpha}) > 0$ for all $\alpha \geq \lambda$, it follows that

$$(3.7) \quad \begin{aligned} & \beta_0(y) \limsup_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] \\ & = \limsup_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y))]. \end{aligned}$$

Since $\beta_0(y) > 0$, by (3.6) and (3.7) we have

$$\limsup_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - y \rangle + h(y_{\alpha}) - h(y)] \leq 0.$$

Since T is h -pseudo-monotone, we have

$$\begin{aligned} & \liminf_{\alpha} [\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] \\ & \geq \min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x). \end{aligned}$$

Since $\beta_0(y) > 0$, we have

$$\begin{aligned} & \beta_0(y) [\liminf_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\ & \geq \beta_0(y) [\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)]. \end{aligned}$$

Thus

$$(3.8) \quad \begin{aligned} & \beta_0(y) [\liminf_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle \\ & \geq \beta_0(y) [\min_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)] + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle. \end{aligned}$$

For $t = 1$ we also have $\phi(x, y_{\alpha}) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_{\alpha}) [\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)] + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle \leq 0$$

for all $\alpha \in \Gamma$. Therefore

$$\begin{aligned}
 0 &\geq \liminf_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x)) \\
 &\quad + \sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle] \\
 &\geq \liminf_{\alpha} [\beta_0(y_{\alpha}) (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\
 (3.9) \quad &\quad + \liminf_{\alpha} [\sum_{p \in E^*} \beta_p(y_{\alpha}) \operatorname{Re} \langle p, y_{\alpha} - x \rangle] \\
 &= \beta_0(y) [\liminf_{\alpha} (\min_{w \in T(y_{\alpha})} \operatorname{Re} \langle w, y_{\alpha} - x \rangle + h(y_{\alpha}) - h(x))] \\
 &\quad + \sum_{p \in E^*} \beta_p(y) \operatorname{Re} \langle p, y - x \rangle.
 \end{aligned}$$

Consequently, by (3.8) and (3.9), we have $\phi(x, y) \leq 0$.

Now, the rest of the proof of Step 1 is similar to the proofs in Step 1 of Theorem 2.1 and Theorem 3.1 in [6]. Thus Step 1 is proved.

Step 2. There exists a point $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re} \langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$.

Also the same proof of Step 2 of Theorem 2.1 shows that there exists $\hat{w} \in T(\hat{y})$ such that $\operatorname{Re} \langle \hat{w}, \hat{y} - x \rangle + h(\hat{y}) - h(x) \leq 0$ for all $x \in S(\hat{y})$. $\square$

If X is compact, we obtain the following immediate consequence of Theorem 3.1:

Theorem 3.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \rightarrow \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \rightarrow 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that the set $\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)] > 0\}$ is open in X . Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Note that if the map $S : X \rightarrow 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semicontinuous at y in X , then the set Σ in Theorem 3.1 is always open in X as can be seen in the proof of the following:

Theorem 3.3. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \rightarrow \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Suppose further that there exist a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} \operatorname{Re} \langle w, y - x_0 \rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists $\hat{y} \in K$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re} \langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

Proof. By virtue of Theorem 3.1, we need only show that the set $\Sigma := \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$ is open in X .

Now, following the same arguments as in the proofs of Theorem 3.2 in [6] and Theorem 2.3, we can similarly show that the set Σ is open in X . Hence by Theorem 3.1 the conclusion follows. $\square$

If X is compact, we obtain the following immediate consequence of Theorem 3.3:

Theorem 3.4. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \rightarrow \mathbb{R}$ be convex such that $h(X)$ is bounded. Let $S : X \rightarrow 2^X$ be continuous such that each $S(x)$ is closed convex and $T : X \rightarrow 2^{E^*}$ be h -pseudo-monotone and be upper semicontinuous from $\operatorname{co}(A)$ to the weak*-topology on E^* for each $A \in \mathcal{F}(X)$ such that each $T(x)$ is weak*-compact convex and $T(X)$ is strongly bounded. Suppose that for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} \operatorname{Re}\langle w, y - x \rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at y from the relative topology on X to the strong topology on E^* . Then there exists $\hat{y} \in X$ such that (i) $\hat{y} \in S(\hat{y})$ and (ii) there exists $\hat{w} \in T(\hat{y})$ with $\operatorname{Re}\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.*

We remark here that in Theorems 3.1-3.4, the condition “ $h : E \rightarrow \mathbb{R}$ be convex” can be replaced by the condition “ $h : X \rightarrow \mathbb{R}$ be convex such that $h|_{\operatorname{co}(A)}$ is continuous for each $A \in \mathcal{F}(X)$ ”.

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