A THEOREM ON ITERATIONS OF POLYNOMIAL MAPS IN SEVERAL VARIABLES

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ABSTRACT. We establish a polynomial version of a theorem obtained by Enflo, Gurarii, Lomonosov and Lyubich for linear operators. As a consequence, we also derive a polynomial version of a result due to Pták.

1. INTRODUCTION

Throughout this paper $\mathbf{K}$ denotes either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, $\mathbf{N}$ denotes the set of non-negative integers and $\mathbf{N}^* = \mathbf{N} - \{0\}$. We shall always assume $\mathbf{K}^n$ ($n \in \mathbf{N}^*$) endowed with its euclidean norm, and if $P : \mathbf{K}^n \rightarrow \mathbf{K}^n$ is a homogeneous polynomial map we consider

$$\|P\| = \sup_{\|x\| \leq 1} \|Px\|.$$

If X and Y are sets, then the relation “ $X \supset Y$ ” means that X contains Y *properly* and the relation “ $X \supseteq Y$ ” means that X contains or is equal to Y . Moreover, we denote by $X - Y$ the difference set $\{x \in X; x \notin Y\}$.

Pták [4] established the following result:

If $T : \mathbf{K}^n \rightarrow \mathbf{K}^n$ is a linear operator which satisfies $1 = \|T\| = \dots = \|T^n\|$, then $\|T^j\| = 1$ for all $j \geq 1$.

Later, Enflo, Gurarii, Lomonosov and Lyubich [1] obtained the following point-wise version of Pták’s result :

There is a constant $C = C_{\mathbf{K}}(n) \in \mathbf{N}^$ such that for any linear operator $T : \mathbf{K}^n \rightarrow \mathbf{K}^n$ and any $x \in \mathbf{K}^n$, the relation $1 = \|x\| = \|Tx\| = \dots = \|T^C x\|$ implies $\|T^j x\| = 1$ for all $j \geq 1$.*

The goal of the present paper is to obtain polynomial versions of the above mentioned theorems. Our main result is the following:

Theorem 1. *Let n and d be positive integers. Then there is a constant $M = M_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial map $P : \mathbf{K}^n \rightarrow \mathbf{K}^n$ of degree d and any $x \in \mathbf{K}^n$, the relation*

$$(1) \quad 1 = \|x\| = \|Px\| = \dots = \|P^M x\|$$

implies

$$(2) \quad \|P^j x\| = 1 \text{ for all } j \geq 1.$$

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As an immediate consequence of this theorem, we have the following

Theorem 2. *Let n and d be positive integers. Then there is a constant $N = N_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial map $P : \mathbf{K}^n \rightarrow \mathbf{K}^n$ of degree d , the relation $1 = \|P\| = \dots = \|P^N\|$ implies $\|P^j\| = 1$ for all $j \geq 1$.*

In the next section we prove theorem 1 modulo two lemmas. These lemmas will be proved in §3. Finally, in §4 we show that the argument used to prove theorem 1 can also be applied to establish other results of similar type.

2. PROOF OF THEOREM 1

Without loss of generality, we may assume $\mathbf{K} = \mathbf{R}$. Put

$$\alpha_j = 2^j d^{1+2+\dots+j} \quad (j \geq 1).$$

We consider a set B of the form

$$B = \{b_{i_0, \dots, i_r}; r \in \mathbf{N} \text{ and } i_0, \dots, i_r \in \mathbf{N}^*\},$$

where the $b_{i_0, \dots, i_r}$'s are distinct objects. The elements of B will be called *pieces*. We define a relation " $>$ " on B by

$$b_{i_0, \dots, i_r} > b_{j_0, \dots, j_s} \text{ if and only if } s = r + 1 \text{ and } j_0 = i_0, \dots, j_r = i_r.$$

If $b, b' \in B$ and $b > b'$, we say that b' is a *subpiece* of b . We also consider the collection Ω of all finite sequences

$$S = (B_1, \dots, B_r)$$

of subsets of B with the following properties :

- (a) $B_1 \subset \{b_i; i \in \mathbf{N}^*\}$;
- (b) Each B_j has at most α_j pieces;
- (c) For each $j \in \{2, \dots, r\}$ and for each piece x of B_j , either x is a piece of B_{j-1} or x is a subpiece of some piece y of B_{j-1} ; moreover, in the latter case, we cannot have $y \in B_j$;
- (d) $B_1 \neq B_2 \neq \dots \neq B_r$;
- (e) There cannot exist integers $1 \leq i_0 < \dots < i_n \leq r$ and pieces $x_{i_0} \in B_{i_0}, \dots, x_{i_n} \in B_{i_n}$ so that $x_{i_0} > x_{i_1} > \dots > x_{i_n}$.

We claim that the following holds :

- (3) There is a constant $M = M(n, d) \in \mathbf{N}^*$ such that every sequence in Ω has length $\leq M$, where the length of a sequence $S = (B_1, \dots, B_r) \in \Omega$ is defined to be the number r .

Assume (3) for the moment, and let us see how it can be used to prove theorem

- 1. For each $j \in \mathbf{N}^*$, let q_j be the real homogeneous polynomial given by

$$q_j(y) = \langle P^j y, P^j y \rangle - \langle y, y \rangle^{d^j} \quad (y \in \mathbf{R}^n),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^n$. Define

$$\begin{aligned} A_j &= \{y \in \mathbf{R}^n; q_k(y) = 0 \text{ for } k = 1, \dots, j\} \\ &= \{y \in \mathbf{R}^n; \|P^k y\| = \|y\|^{d^k} \text{ for } k = 1, \dots, j\}, \end{aligned}$$

for $j \geq 1$. Obviously,

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$

For each $j \in \mathbf{N}^*$, consider the algebraic subset

$$Y_j = Z(\tilde{q}_1) \cap \dots \cap Z(\tilde{q}_j)$$

of the projective $(n-1)$ -space $\mathbf{P}^{n-1}$ over $\mathbf{C}$ (where $\tilde{q}_j$ denotes the polynomial q_j regarded as a complex polynomial in n complex variables and $Z(\tilde{q}_j)$ is the zero set of $\tilde{q}_j$ in $\mathbf{P}^{n-1}$; see [3]). Obviously, $Y_1 \supseteq Y_2 \supseteq \dots$. Since $\mathbf{P}^{n-1}$ is a Noetherian topological space, there is a greatest integer γ such that

$$Y_1 \supset Y_2 \supset \dots \supset Y_\gamma.$$

Now we need the following:

Lemma 3. *There is a sequence in Ω with length γ .*

The proofs of this and the next lemma will be left to the next section.

By (3) and lemma 3, $\gamma \leq M$. Our definition of γ implies that $Y_\gamma = Y_{\gamma+1}$. Hence, if $y \in A_\gamma - \{0\}$ then

$$\phi(y) \in Z(\tilde{q}_1) \cap \dots \cap Z(\tilde{q}_\gamma) = Y_\gamma = Y_{\gamma+1} \subseteq Z(\tilde{q}_{\gamma+1})$$

(where $\phi(y)$ denotes the equivalence class of y in $\mathbf{P}^{n-1}$), which implies $y \in A_{\gamma+1}$. Thus, $A_\gamma = A_{\gamma+1}$. Now, if $y \in A_{\gamma+1}$ then $Py \in A_\gamma$ and so $Py \in A_{\gamma+1}$, which implies $y \in A_{\gamma+2}$. Hence, $A_{\gamma+1} = A_{\gamma+2}$. By induction, we see that $A_\gamma = A_{\gamma+1} = A_{\gamma+2} = \dots$. In particular,

$$A_M = A_{M+1} = A_{M+2} = \dots$$

If x satisfies (1), then $x \in A_M$ and so $x \in A_j$ for all $j \geq M$, which gives (2). This proves theorem 1.

We shall establish (3) by presenting an algorithm that constructs a longest possible sequence in Ω . The main motivation for the algorithm is based on the following observation: By (c), we can think of B_j as obtained from B_{j-1} by repeating or deleting or replacing by subpieces each of the pieces of B_{j-1} .

First, let us make some definitions. Let $S = (B_1, \dots, B_r) \in \Omega$. If $x \in B_j$ for some $j \in \{1, \dots, r\}$, say $x = b_{j_0, \dots, j_m}$, we define the depth of x (denoted $\text{depth}(x)$) as being the number m . Note that this is the largest integer with the property that we can find integers $1 \leq i_0 < \dots < i_m = j$ and pieces $x_{i_0} \in B_{i_0}, \dots, x_{i_m} \in B_{i_m}$ so that

$$x_{i_0} > x_{i_1} > \dots > x_{i_m} = x.$$

With this definition we see that condition (e) merely says that each piece in each B_j must have depth $\leq n-1$. We also define the numbers

$$N_i(S, j) = \text{card}\{x \in B_j; \text{depth}(x) = i\} \quad (i = 0, \dots, n-1; j = 1, \dots, r)$$

and put

$$N(S, j) = (N_0(S, j), \dots, N_{n-1}(S, j)) \quad (j = 1, \dots, r).$$

Note that $N(S, j) \in \mathbf{N}^n$. In the sequel we consider $\mathbf{N}^n$ endowed with its lexicographic order relation, which we denote by " $\leq$ ".

Algorithm. 1. Put $\tilde{B}_1 = \{b_1, \dots, b_{\alpha_1}\}$.

2. While $\tilde{B}_j \neq \emptyset$ do:

- if $\tilde{B}_j$ has a piece of depth $n-1$, choose one such piece x and define $\tilde{B}_{j+1} = \tilde{B}_j - \{x\}$;

- if $\tilde{B}_j$ has no piece of depth $n-1$, choose a piece x of $\tilde{B}_j$ with maximum depth. Then choose $\alpha_{j+1} - \text{card}(\tilde{B}_j) + 1$ subpieces $x_1, \dots, x_{\alpha_{j+1} - \text{card}(\tilde{B}_j) + 1}$ of x and define $\tilde{B}_{j+1} = (\tilde{B}_j - \{x\}) \cup \{x_1, \dots, x_{\alpha_{j+1} - \text{card}(\tilde{B}_j) + 1}\}$.
3. If $\tilde{B}_j = \emptyset$, stop.

Let $\tilde{S} = (\tilde{B}_1, \tilde{B}_2, \dots)$ be the sequence generated by our algorithm. Clearly our algorithm never generates pieces of depth $\geq n$, and in each step it decreases the depth numbers in lexicographic order: $N(\tilde{S}, j+1) < N(\tilde{S}, j)$. Since the lexicographic order relation on $\mathbf{N}^n$ is a well-order, it follows that our algorithm must end in a finite number of steps, so that $\tilde{S}$ is a finite sequence; say

$$\tilde{S} = (\tilde{B}_1, \dots, \tilde{B}_M).$$

Moreover, it is immediate to check that $\tilde{S} \in \Omega$. Now, we will need the fact that our sequence $\tilde{S}$ has “maximal depth numbers”:

Lemma 4. *If $S = (B_1, \dots, B_r)$ is any element of Ω , then*

$$N(\tilde{S}, j) \geq N(S, j) \quad \text{for all } j = 1, \dots, \min\{M, r\}.$$

Let us finally prove (3). We claim that for any sequence $S = (B_1, \dots, B_r) \in \Omega$, we have $r \leq M$. Suppose that this is not the case, and let $S \in \Omega$ be a sequence with length $r > M$. By lemma 4, $N(S, M) \leq N(\tilde{S}, M)$. Since $\tilde{B}_M = \emptyset$, we have $N(\tilde{S}, M) = (0, \dots, 0)$. Hence, $N(S, M) = (0, \dots, 0)$, which implies $B_M = \emptyset$. It now follows from (c) that $B_{M+1} = \emptyset$. Hence, $B_M = B_{M+1}$, which contradicts (d).

3. PROOFS OF THE LEMMAS

Proof of lemma 3. For each $j \in \{1, \dots, \gamma\}$, let C_j be the set of all irreducible components of Y_j . Since each $\tilde{q}_j$ has degree $\leq 2d^j$, it follows that

$$\text{card}(C_j) \leq (2d)(2d^2) \dots (2d^j) = \alpha_j \quad \text{for } j = 1, \dots, \gamma$$

([2], example 8.4.6). For each $j \in \{1, \dots, \gamma\}$ and each $Z \in C_j$, we define the depth of Z (denoted $\text{depth}(Z)$) as the greatest integer $m \in \mathbf{N}$ such that we can find integers $1 \leq i_0 < \dots < i_m = j$ and elements $Z_{i_0} \in C_{i_0}, \dots, Z_{i_m} \in C_{i_m}$ so that

$$Z_{i_0} \supset Z_{i_1} \supset \dots \supset Z_{i_m} = Z.$$

Note that since $\mathbf{P}^{n-1}$ has dimension $n-1$ [3], each element of each C_j must have depth $\leq n-1$. We shall construct, inductively, subsets $B_1, \dots, B_\gamma$ of B and bijections $\phi_1 : C_1 \rightarrow B_1, \dots, \phi_\gamma : C_\gamma \rightarrow B_\gamma$, so that

$$(4) \quad (B_1, \dots, B_j) \in \Omega$$

and

$$(5) \quad \text{depth}(\phi_j(Z)) \leq \text{depth}(Z) \quad \text{for all } Z \in C_j,$$

for $j = 1, \dots, \gamma$. This will prove the lemma.

We begin by putting $B_1 = \{b_1, \dots, b_{\text{card}(C_1)}\}$ and by choosing an arbitrary bijection $\phi_1 : C_1 \rightarrow B_1$. It is clear that (4) and (5) hold for $j = 1$. Suppose that for some $t \in \{1, \dots, \gamma-1\}$ we have already constructed $B_1, \dots, B_t \subset B$ and bijections $\phi_1 : C_1 \rightarrow B_1, \dots, \phi_t : C_t \rightarrow B_t$ so that (4) and (5) hold for $j = 1, \dots, t$. Since $Y_t \supset Y_{t+1}$, each element of C_{t+1} is either an element of C_t or a proper subset of some element of C_t . Let $Z_1, \dots, Z_\beta$ be the elements of $C_{t+1} \cap C_t$. We define

$$\phi_{t+1}(Z_i) = \phi_t(Z_i) \quad \text{for } i = 1, \dots, \beta.$$

Let $W_1, \dots, W_\alpha$ be an enumeration of the elements of $C_t - C_{t+1}$. Let $W_{1,1}, \dots, W_{1,\beta_1}$ be the elements of C_{t+1} that are proper subsets of W_1 . By definition, $\phi_t(W_1) \in B$, say $\phi_t(W_1) = b_{i_1, \dots, i_s}$. We then define

$$\phi_{t+1}(W_{1,i}) = b_{i_1, \dots, i_s, i} \quad \text{for } i = 1, \dots, \beta_1.$$

Now, let $W_{2,1}, \dots, W_{2,\beta_2}$ be the elements of $C_{t+1} - \{W_{1,1}, \dots, W_{1,\beta_1}\}$ that are proper subsets of W_2 . If $\phi_t(W_2) = b_{j_1, \dots, j_w}$, we then define

$$\phi_{t+1}(W_{2,i}) = b_{j_1, \dots, j_w, i} \quad \text{for } i = 1, \dots, \beta_2.$$

By continuing this process, we obtain a function $\phi_{t+1} : C_{t+1} \rightarrow B$. Put

$$B_{t+1} = \phi_{t+1}(C_{t+1}).$$

The main ingredient for checking that B_{t+1} and ϕ_{t+1} have the desired properties is the fact that for any $S' = (B'_1, \dots, B'_r) \in \Omega$ no two pieces of any B'_j can be comparable under “ $>$ ” (which can be easily proved by induction on j). We leave the details to the reader.

Proof of lemma 4. We proceed by induction on j . If $j = 1$ then

$$N(\tilde{S}, 1) = (\alpha_1, 0, 0, \dots, 0) \geq N(S, 1).$$

Suppose that for a certain $j \in \{1, \dots, \min\{M, r\} - 1\}$ we have $N(\tilde{S}, j) \geq N(S, j)$. We have to show that

$$(6) \quad N(\tilde{S}, j+1) \geq N(S, j+1).$$

We have two cases:

Case 1. $N(\tilde{S}, j) = N(S, j)$: This means that

$$N_i(\tilde{S}, j) = N_i(S, j) \quad \text{for } i = 0, \dots, n-1.$$

Let β be the maximum depth of the pieces of $\tilde{B}_j$, so that

$$N_{\beta+1}(\tilde{S}, j) = \dots = N_{n-1}(\tilde{S}, j) = 0.$$

By our construction of $\tilde{B}_{j+1}$,

$$(7) \quad N_i(\tilde{S}, j+1) = N_i(\tilde{S}, j) = N_i(S, j) \quad \text{for } i = 0, \dots, \beta-1.$$

Suppose that for some $i \in \{1, \dots, \beta\}$ we have $N_i(S, j+1) > N_i(S, j)$. Then, in the construction of B_{j+1} , at least one piece of B_j with depth $i-1$ must have been replaced by subpieces. Let $i_0 \in \{0, \dots, \beta-1\}$ be the smallest integer such that some piece of B_j with depth i_0 was replaced by subpieces. Then, $N_i(\tilde{S}, j+1) = N_i(S, j) \geq N_i(S, j+1)$ for $i = 0, \dots, i_0-1$, and $N_{i_0}(\tilde{S}, j+1) = N_{i_0}(S, j) > N_{i_0}(S, j+1)$ (by (7)), and so (6) holds. Thus, assume that

$$N_i(S, j+1) \leq N_i(S, j) \quad \text{for } i = 1, \dots, \beta.$$

If for some $i \in \{0, \dots, \beta-1\}$ we have $N_i(S, j+1) < N_i(S, j)$, then (6) holds (by (7)). Hence, assume

$$(8) \quad N_i(S, j+1) = N_i(S, j) \quad \text{for } i = 0, \dots, \beta-1.$$

By (7) and (8), we obtain

$$(9) \quad N_i(\tilde{S}, j+1) = N_i(S, j+1) \quad \text{for } i = 0, \dots, \beta-1.$$

By (8), in the construction of B_{j+1} , no piece of B_j with depth $\leq \beta-1$ was removed nor replaced by subpieces. Since $N_i(S, j) = N_i(\tilde{S}, j) = 0$ for $j = \beta+1, \dots, n-1$,

some piece of B_j with depth β must have been removed or replaced by subpieces, so that $N_\beta(S, j+1) \leq N_\beta(S, j) - 1 = N_\beta(\tilde{S}, j) - 1 = N_\beta(\tilde{S}, j+1)$. If $N_\beta(S, j+1) < N_\beta(\tilde{S}, j+1)$, then (6) follows from (9). Thus, assume

$$(10) \quad N_\beta(\tilde{S}, j+1) = N_\beta(S, j+1).$$

If $\beta = n-1$, (9) and (10) give equality in (6). Suppose $\beta < n-1$. By our construction of $\tilde{B}_{j+1}$, we have

$$(11) \quad N_{\beta+1}(\tilde{S}, j+1) = \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \dots + N_\beta(\tilde{S}, j+1))$$

and

$$(12) \quad N_{\beta+2}(\tilde{S}, j+1) = \dots = N_{n-1}(\tilde{S}, j+1) = 0.$$

Since $\text{card}(B_{j+1}) \leq \alpha_{j+1}$,

$$\begin{aligned} N_{\beta+1}(S, j+1) &\leq \alpha_{j+1} - (N_0(S, j+1) + \dots + N_\beta(S, j+1)) \\ &= \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \dots + N_\beta(\tilde{S}, j+1)) = N_{\beta+1}(\tilde{S}, j+1) \end{aligned}$$

(by (11)). Again, if $N_{\beta+1}(S, j+1) < N_{\beta+1}(\tilde{S}, j+1)$, then (6) follows from (9) and (10). So, suppose

$$N_{\beta+1}(S, j+1) = N_{\beta+1}(\tilde{S}, j+1).$$

Then $N_0(S, j+1) + \dots + N_{\beta+1}(S, j+1) = N_0(\tilde{S}, j+1) + \dots + N_{\beta+1}(\tilde{S}, j+1) = \alpha_{j+1}$, and therefore

$$N_{\beta+2}(S, j+1) = \dots = N_{n-1}(S, j+1) = 0.$$

In view of (12), we conclude that we have equality in (6).

Case 2. $N(\tilde{S}, j) > N(S, j)$: In this case we shall show that

$$(13) \quad N(\tilde{S}, j+1) > N(S, j+1).$$

Our hypothesis implies that there is an $\alpha \in \{0, \dots, n-1\}$ such that

$$N_0(\tilde{S}, j) = N_0(S, j), \dots, N_{\alpha-1}(\tilde{S}, j) = N_{\alpha-1}(S, j) \quad \text{and} \quad N_\alpha(\tilde{S}, j) > N_\alpha(S, j).$$

In particular, $N_\alpha(\tilde{S}, j) \neq 0$, and therefore

$$(14) \quad N_i(\tilde{S}, j+1) = N_i(\tilde{S}, j) = N_i(S, j) \quad \text{for } i = 0, \dots, \alpha-1.$$

Suppose that for some $i \in \{1, \dots, \alpha\}$ we have $N_i(S, j+1) > N_i(S, j)$. Then, in the construction of B_{j+1} , at least one piece of B_j with depth $i-1$ must have been replaced by subpieces. Let $i_0 \in \{0, \dots, \alpha-1\}$ be the smallest integer such that some piece of B_j with depth i_0 was replaced by subpieces. Then, $N_i(\tilde{S}, j+1) = N_i(S, j) \geq N_i(S, j+1)$ for $i = 0, \dots, i_0-1$, and $N_{i_0}(\tilde{S}, j+1) = N_{i_0}(S, j) > N_{i_0}(S, j+1)$ (by (14)), and so (13) holds. Thus, assume that

$$(15) \quad N_i(S, j+1) \leq N_i(S, j) \quad \text{for } i = 1, \dots, \alpha.$$

If for some $i \in \{0, \dots, \alpha-1\}$ we have $N_i(S, j+1) < N_i(S, j)$, then (13) holds (by (14) and (15)). Hence, assume

$$(16) \quad N_i(S, j+1) = N_i(S, j) \quad \text{for } i = 0, \dots, \alpha-1.$$

By (14) and (16),

$$(17) \quad N_i(\tilde{S}, j+1) = N_i(S, j+1) \quad \text{for } i = 0, \dots, \alpha-1.$$

Now, if some piece of $\tilde{B}_j$ has depth $> \alpha$, then

$$N_\alpha(\tilde{S}, j+1) = N_\alpha(\tilde{S}, j) > N_\alpha(S, j) \geq N_\alpha(S, j+1)$$

(by (15)), and so (13) follows from (17). Suppose that the maximum depth of the pieces in $\tilde{B}_j$ is exactly equal to α . Then

$$N_\alpha(\tilde{S}, j+1) = N_\alpha(\tilde{S}, j) - 1 \geq N_\alpha(S, j) \geq N_\alpha(S, j+1).$$

If $N_\alpha(\tilde{S}, j+1) > N_\alpha(S, j+1)$, then (13) follows from (17). So, assume

$$(18) \quad N_\alpha(\tilde{S}, j+1) = N_\alpha(S, j+1).$$

Hence, $N_\alpha(S, j) = N_\alpha(S, j+1)$, and so we conclude from (16) that $\alpha < n-1$ and that

$$N_{\alpha+1}(S, j+1) \leq N_{\alpha+1}(S, j).$$

Now, the fact that $\tilde{B}_j$ has no piece with depth $> \alpha$ implies that

$$N_{\alpha+1}(\tilde{S}, j+1) = \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \dots + N_\alpha(\tilde{S}, j+1)).$$

Consequently,

$$\begin{aligned} N_{\alpha+1}(S, j+1) &\leq N_{\alpha+1}(S, j) \leq \alpha_j - (N_0(S, j) + \dots + N_\alpha(S, j)) \\ &< \alpha_{j+1} - (N_0(\tilde{S}, j+1) + \dots + N_\alpha(\tilde{S}, j+1)) = N_{\alpha+1}(\tilde{S}, j+1). \end{aligned}$$

In view of (17) and (18), we conclude that (13) holds. This completes the proof of lemma 4.

Remark 5. The above proof shows that if $S = (B_1, \dots, B_r) \in \Omega$ and if $N(\tilde{S}, j) > N(S, j)$ for some $j \in \{0, \dots, \min\{M, r\}\}$, then

$$N(\tilde{S}, t) > N(S, t) \quad \text{for all } t \in \{j, \dots, \min\{M, r\}\}.$$

In particular, $r < M$. This shows that the sequences $S \in \Omega$ that have length M are exactly those that satisfy the condition

$$N(S, j) = N(\tilde{S}, j) \quad \text{for all } j \in \{0, \dots, M\}.$$

4. FURTHER RESULTS

We now remark that the argument used to prove theorem 1 can also be applied to establish other results of similar type. For instance, we have the following

Theorem 6. *Let n and d be positive integers. Then there is a constant $L = L_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial maps $P, Q : \mathbf{K}^n \rightarrow \mathbf{K}^n$ of degree d and any $x, y \in \mathbf{K}^n$, the relation*

$$\|P^j x\| = \|Q^j y\| \quad \text{for } j = 1, \dots, L$$

implies

$$\|P^j x\| = \|Q^j y\| \quad \text{for all } j \geq 1.$$

Proof. We may assume $\mathbf{K} = \mathbf{R}$. Now, we have to consider the homogeneous polynomial $q_j : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ given by

$$q_j(z, w) = \langle P^j z, P^j z \rangle - \langle Q^j w, Q^j w \rangle \quad (z, w \in \mathbf{R}^n).$$

As before, we define

$$\begin{aligned} A_j &= \{(z, w) \in \mathbf{R}^{2n}; q_k(z, w) = 0 \text{ for } k = 1, \dots, j\} \\ &= \{(z, w) \in \mathbf{R}^{2n}; \|P^k z\| = \|Q^k w\| \text{ for } k = 1, \dots, j\} \end{aligned}$$

and consider the algebraic subset

$$Y_j = Z(\tilde{q}_1) \cap \dots \cap Z(\tilde{q}_j)$$

of $\mathbf{P}^{2n-1}$. The same argument used to prove theorem 1 then applies (the α_j 's are the same, but we have to consider $2n$ in place of n).

Similarly, we obtain

Theorem 7. *Let n and d be positive integers. Then there is a constant $R = R_{\mathbf{K}}(n, d) \in \mathbf{N}^*$ such that for any homogeneous polynomial maps $P, Q : \mathbf{K}^n \rightarrow \mathbf{K}^n$ of degree d and any $x, y \in \mathbf{K}^n$, the relation*

$$1 = \|x - y\| = \|Px - Qy\| = \dots = \|P^R x - Q^R y\|$$

implies

$$\|P^j x - Q^j y\| = 1 \text{ for all } j \geq 1.$$

Remark 8. Note that theorem 7 generalizes theorem 1 (just set $Q = P$ and $y = 0$).

We close the paper by proposing the following question:

Open problem. What are the best values of the constants $M_{\mathbf{R}}(n, d)$, $M_{\mathbf{C}}(n, d)$, $N_{\mathbf{R}}(n, d)$ and $N_{\mathbf{C}}(n, d)$?

Only the following are known: $N_{\mathbf{R}}(n, 1) = N_{\mathbf{C}}(n, 1) = n$ (see [4]), $M_{\mathbf{R}}(1, 1) = M_{\mathbf{C}}(1, 1) = 1$, $M_{\mathbf{R}}(2, 1) = 3$, $M_{\mathbf{C}}(2, 1) = 4$ and $M_{\mathbf{C}}(3, 1) = 8$ (see [1]). Moreover, some estimates for $M_{\mathbf{R}}(n, 1)$ and $M_{\mathbf{C}}(n, 1)$ were obtained in [1].

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