STIELTJES MOMENT SEQUENCES AND POSITIVE DEFINITE MATRIX SEQUENCES

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ABSTRACT. For a certain constant $\delta > 0$ (a little less than $1/4$), every function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfying $f(n)^2 \leq \delta f(n-1)f(n+1)$, $n \in \mathbb{N}$, is a Stieltjes indeterminate Stieltjes moment sequence. For every indeterminate moment sequence $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ there is a positive definite matrix sequence (a_n) which is not of positive type and which satisfies $\text{tr}(a_{n+2}) = f(n)$, $n \in \mathbb{N}_0$. For a certain constant $\varepsilon > 0$ (a little greater than $1/6$), for every function $\varphi: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfying $\varphi(n)^2 \leq \varepsilon \varphi(n-1)\varphi(n+1)$, $n \in \mathbb{N}$, there is a convolution semigroup $(\mu_t)_{t \geq 0}$ of measures on $\mathbb{R}_+$, with moments of all orders, such that $\varphi(n) = \int x^n d\mu_1(x)$, $n \in \mathbb{N}_0$, and for every such convolution semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for all $t > 0$.

1. INTRODUCTION

It was shown by Boas [4] that every function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ of sufficiently rapid growth is a Stieltjes indeterminate Stieltjes moment sequence. More precisely, he proved that every function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfying

$$(1) \quad f(0) \geq 1, \quad f(n) \geq (nf(n-1))^n \quad \text{for } n \in \mathbb{N}$$

is a Stieltjes moment sequence, and inferred that if f satisfies (1) and if furthermore $f(2) \geq (2f(1) + 2)^2$ then the Stieltjes moment sequence $n \mapsto f(2n)$ is Stieltjes indeterminate. We shall show, for a certain absolute constant δ (a little less than $1/4$), that every function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfying

$$(2) \quad f(n)^2 \leq \delta f(n-1)f(n+1), \quad n \in \mathbb{N},$$

is a Stieltjes indeterminate Stieltjes moment sequence. The fact that every such f is a moment sequence was shown in [3].

It was shown in [2] by an example that there is a positive definite sequence (a_n) in $\mathbb{M}_2$ (the 2×2 complex matrices) which is not of positive type. (Hence there is a positive linear mapping of $\mathbb{C}[x]$ into $\mathbb{M}_2$ which is not completely positive.) In that example,

$$a_{2n} = \begin{pmatrix} 2^{(n+2)!} & 0 \\ 0 & 2^{(n+2)!} \end{pmatrix}, \quad n \geq 2.$$

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We shall show that the same can be achieved with the more modest growth

$$\mathrm{tr}(a_{2n+2}) = \Gamma(c(2n+1))$$

for any $c > 1$.

Finally, for a certain absolute constant ε (a little greater than $1/6$), we shall show that whenever $\varphi: \mathbb{N}_0 \rightarrow]0, \infty[$ is such that

$$(3) \quad \varphi(n)^2 \leq \varepsilon \varphi(n-1) \varphi(n+1), \quad n \in \mathbb{N},$$

then there is a convolution semigroup $(\mu_t)_{t \geq 0}$ of measures on $\mathbb{R}_+$, with moments of all orders, such that

$$\varphi(n) = \int x^n d\mu_1(x), \quad n \in \mathbb{N}_0,$$

and for every such semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for all $t > 0$. This applies to the moment sequence φ of the measure

$$(4) \quad \frac{1}{\sigma\sqrt{2\pi}} e^{-(\log x)^2/2\sigma^2} 1_{]0, \infty[}(x) \frac{dx}{x},$$

provided that $\sigma > 4/3$.

2. MOMENT SEQUENCES AND MATRIX SEQUENCES

A sequence $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ is a *moment sequence* if

$$(5) \quad f(n) = \int x^n d\mu(x), \quad n \in \mathbb{N}_0,$$

for some measure μ on $\mathbb{R}$ having moments of all orders; a *Stieltjes moment sequence* if (5) holds for some measure μ on $\mathbb{R}_+$. A moment sequence f is *determinate* if there is only one measure μ on $\mathbb{R}$ satisfying (5); otherwise, *indeterminate*. A Stieltjes moment sequence f is *Stieltjes determinate* if there is only one measure μ on $\mathbb{R}_+$ satisfying (5); otherwise, *Stieltjes indeterminate*. A sequence $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ is a moment sequence if and only if f is *positive definite* in the sense that for each $n \in \mathbb{N}_0$ the matrix $(f(j+k))_{j,k=0}^n$ is positive semidefinite. A sequence $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ is a Stieltjes moment sequence if and only if f and the shifted sequence $n \mapsto f(n+1): \mathbb{N}_0 \rightarrow \mathbb{R}$ are both positive definite.

Definition 1. We denote by δ the positive number satisfying $\sum_{n=1}^{\infty} \delta^{n^2} = 1/4$.

Clearly $\delta < 1/4$. Below we shall obtain the estimate $\delta > 15/61$. Since (for $0 < x < 1$) $x + x^4 < \sum_{n=1}^{\infty} x^{n^2} < x + x^4 + \sum_{n=3}^{\infty} x^{7n-12} = x + x^4 + x^9/(1-x^7)$, one can show $x < \delta < y$ by verifying $x + x^4 + x^9/(1-x^7) < 1/4 < y + y^4$. In this way, using a calculator, one gets $0.246315 < \delta < 0.246319$.

Theorem 1. Every function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfying (2) is a Stieltjes indeterminate Stieltjes moment sequence.

Proof. We shall show here that every such function f is a Stieltjes moment sequence. This is used in the proof of Theorem 2 below, from which it then follows that every such f is Stieltjes indeterminate.

If f satisfies (2), the shifted sequence $n \mapsto f(n+1)$ satisfies the corresponding set of inequalities. It therefore suffices to show that every such f is positive definite. The positive definiteness of such f was obtained in [3] as a corollary of a difficult theorem on positive definite kernels. We present here a simplified proof.

Suppose $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfies (2). We shall show that f is strictly positive definite in the sense that for each $n \in \mathbb{N}_0$ the matrix $(f(j+k))_{j,k=0}^n$ is positive definite, that is, the determinants

$$D_n = \begin{vmatrix} f(0) & \dots & f(n) \\ \vdots & & \vdots \\ f(n) & \dots & f(2n) \end{vmatrix}$$

are positive. The inequalities (2) are equivalent to saying that the function $\psi: \mathbb{N}_0 \rightarrow]0, \infty[$ defined by $\psi(n) = f(n)\delta^{n^2/2}$ is log convex. It follows that if $s \in \mathbb{N}_0$ and $j, k \in \{0, \dots, s\}$ then $\psi(j+s)\psi(k+s) \leq \psi(j+k)\psi(2s)$, that is,

$$(6) \quad \frac{f(j+s)f(k+s)}{f(j+k)f(2s)} \leq \delta^{(s-j)(s-k)}.$$

Keeping n fixed, by backwards induction on m we shall define $u_{j,k}(m)$ for $m = n, n-1, \dots, 0$ and $j, k \in \{0, \dots, m\}$ such that

$$(7) \quad u_{j,k}(n) = 1 \quad \text{for all } j, k,$$

$$(8) \quad \frac{1}{2} \leq 1 - 2 \sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)} \leq u_{j,k}(m) \leq 1 \quad \text{for all } m, j, k,$$

$$u_{j,k}(m) = u_{j,k}(m+1) - \frac{u_{j,m+1}(m+1)u_{m+1,k}(m+1)}{u_{m+1,m+1}(m+1)} \\ \times \frac{f(j+m+1)f(m+1+k)}{f(j+k)f(2m+2)}$$

$$(9) \quad \text{for } m < n \text{ and all } j, k.$$

The inequality to the extreme left in (8) always holds for $j, k \in \{0, \dots, m\}$ since

$$\sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)} \leq \sum_{s=m+1}^{\infty} \delta^{(s-m)^2} = \sum_{p=1}^{\infty} \delta^{p^2} = \frac{1}{4}.$$

Clearly (8) is fulfilled for $m = n$. Let $m < n$ and suppose that $u_{j,k}(m+1)$ is defined for all j and k and satisfies the condition (8*) obtained by taking $m+1$ instead of m in (8). By (8*) we have $u_{j,k}(m+1) > 0$ for all j and k , which shows that $u_{j,k}(m)$ is well defined by (9) and satisfies $u_{j,k}(m) \leq u_{j,k}(m+1) \leq 1$ (by (8*)). By (6), applied to $s = m+1$, and by (8*),

$$u_{j,k}(m) \geq 1 - 2 \sum_{s=m+2}^{\infty} \delta^{(s-j)(s-k)} - 2\delta^{(m+1-j)(m+1-k)} = 1 - 2 \sum_{s=m+1}^{\infty} \delta^{(s-j)(s-k)},$$

completing the definition of the $u_{j,k}(m)$ and the proof that they have the properties stated. We now define

$$E_m = \det \left((u_{j,k}(m)f(j+k))_{j,k=0}^m \right), \quad m = 0, \dots, n.$$

Here $E_0 = u_{0,0}(0)f(0) > 0$. Supposing that $m < n$ and that $E_m > 0$, using row operations to get zeroes in the last column of E_{m+1} except in the last entry, we get $E_{m+1} = u_{m+1,m+1}(m+1)f(2m+2)E_m > 0$. In particular, $D_n = E_n > 0$. $\square$

It is shown in [3] that for $x > 1/4$, a function $f: \mathbb{N}_0 \rightarrow]0, \infty[$ may satisfy the inequalities obtained by replacing δ by x in (2), yet not be positive definite.

Remark. Let us call a Stieltjes moment sequence $f: \mathbb{N}_0 \rightarrow]0, \infty[$ *log convex stable* if fg is a Stieltjes moment sequence for every log convex function $g: \mathbb{N}_0 \rightarrow]0, \infty[$. We have seen that $n \mapsto \gamma^{-n^2/2}$ is log convex stable if $\gamma \leq \delta$, but not if $\gamma > 1/4$. Suppose f is log convex stable. It can be shown (using ideas from the Remark following Corollary 1 in [3]) that the kernel

$$\begin{pmatrix} f(0) & f(1) & & & \\ f(1) & f(2) & f(3) & & \\ & f(3) & f(4) & \ddots & \\ & & & \ddots & \ddots \end{pmatrix}$$

(with zeroes everywhere except in the main diagonal and the two adjacent ones) is positive definite. (Similarly with $f(n+1)$ instead of $f(n)$.) It follows that the numbers

$$\delta_n = \frac{f(2n+1)^2}{f(2n)f(2n+2)}, \quad n \in \mathbb{N}_0,$$

have the property that the numbers D_n defined by

$$D_{-1} = D_0 = 1, \quad D_n = D_{n-1} - \delta_{n-1}D_{n-2} \quad (n \in \mathbb{N})$$

are all positive. This implies that there are limits to how many times in a row, and by how much, the δ_n can exceed $1/4$. In fact, it can be shown that if k consecutive δ_n are $\geq x > 1/4$ then

$$k+1 < \frac{\pi}{\arctan \sqrt{4x-1}}.$$

A sequence $(a_n)_{n \in \mathbb{N}_0}$ in $\mathbb{M}_2$ is *positive definite* if for each $\xi \in \mathbb{C}^2$ the scalar sequence $(\langle a_n \xi, \xi \rangle)_{n \in \mathbb{N}_0}$ is positive definite. The sequence (a_n) is of *positive type* if $\sum_{j,k=0}^n \langle a_{j+k} \xi_j, \xi_k \rangle \geq 0$ for all $n \in \mathbb{N}_0$ and all $\xi_0, \dots, \xi_n \in \mathbb{C}^2$.

Theorem 2. *If $f: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfies (2), there is a sequence $(a_n)_{n \in \mathbb{N}_0}$ in $\mathbb{M}_2$ such that the sequences (a_n) and (a_{n+1}) are positive definite, (a_n) is not of positive type, and $\text{tr}(a_{n+2}) = f(n)$ for all $n \in \mathbb{N}_0$. It follows that f is Stieltjes indeterminate.*

Proof. Put $\alpha = f(0)^2/\delta f(1)$, choose $g > \alpha$, choose $c > g^2/\delta f(0)$, choose $\varphi > \sqrt{cf(0)}$, let $d = \alpha + \varphi^2/(g - \alpha)$, define $h > 0$ by $(g + \varphi/h)^2 = \delta cf(0)$, and put $b = (d + \varphi h)^2/\delta f(0)$. Now define (a_n) by

$$a_0 = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \quad a_1 = \begin{pmatrix} d & \varphi \\ \varphi & g \end{pmatrix}, \quad a_n = \begin{pmatrix} f(n-2) & 0 \\ 0 & f(n-2) \end{pmatrix}$$

for $n \geq 2$. To show that the sequences (a_n) and (a_{n+1}) are positive definite, that is, $(\langle a_n \xi, \xi \rangle)$ is a Stieltjes moment sequence for each $\xi \in \mathbb{C}^2$, by that part of Theorem 1 which was proved above it suffices to show

$$\langle a_n \xi, \xi \rangle^2 \leq \delta \langle a_{n-1} \xi, \xi \rangle \langle a_{n+1} \xi, \xi \rangle$$

for all $\xi \in \mathbb{C}^2$ and $n \geq 1$. For $n \geq 3$, this inequality follows from (2). For $n = 2$, what we have to show is that the smallest eigenvalue of a_1 is at least α . But that eigenvalue is

$$\frac{d+g}{2} - \sqrt{\left(\frac{d-g}{2}\right)^2 + \varphi^2} = \alpha.$$

For $n = 1$ and $\xi = (x, y) \in \mathbb{C}^2$ with $\|\xi\| = 1$, using $2|\operatorname{Re}(x\bar{y})| \leq h|x|^2 + |y|^2/h$,

$$\begin{aligned} & \delta \langle a_0 \xi, \xi \rangle \langle a_2 \xi, \xi \rangle - \langle a_1 \xi, \xi \rangle^2 \\ &= \delta f(0)(b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - (d|x|^2 + g|y|^2 + 2\varphi \operatorname{Re}(x\bar{y}))^2 \\ &\geq \delta f(0)(b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - ((d + \varphi h)|x|^2 + (g + \varphi/h)|y|^2)^2 \\ &= (\delta b f(0) - (d + \varphi h)^2)|x|^4 + (\delta c f(0) - (g + \varphi/h)^2)|y|^4 \\ &\quad + (\delta(b + c)f(0) - 2(d + \varphi h)(g + \varphi/h))|x|^2|y|^2 \\ &\geq (\delta b f(0) - (d + \varphi h)^2)|x|^4 + (\delta c f(0) - (g + \varphi/h)^2)|y|^4 \\ &\quad + (\delta(b + c)f(0) - (d + \varphi h)^2 - (g + \varphi/h)^2)|x|^2|y|^2 \\ &= 0. \end{aligned}$$

To see that (a_n) is not of positive type, note that with $\xi_0 = (0, \varphi)$ and $\xi_1 = (-c, 0)$ we have

$$(10) \quad \sum_{j,k=0}^1 \langle a_{j+k} \xi_j, \xi_k \rangle = c\varphi^2 - 2c\varphi^2 + c^2 f(0) = -c(\varphi^2 - cf(0)) < 0.$$

If for all $\xi, \eta \in \mathbb{C}^2$ the Stieltjes moment sequence $(\langle a_n \xi, \xi \rangle + \langle a_n \eta, \eta \rangle)_{n \in \mathbb{N}_0}$ were Stieltjes determinate, as in the proof of [2], Theorem 2, it would follow that (a_n) were the moment sequence of a measure on $\mathbb{R}_+$ with positive semidefinite matrix values, contradicting the fact that (a_n) is not of positive type. Hence there exist $\xi, \eta \in \mathbb{C}^2$ such that $(\langle a_n \xi, \xi \rangle + \langle a_n \eta, \eta \rangle)$ is Stieltjes indeterminate. Now there exist $\zeta_1, \dots, \zeta_m \in \mathbb{C}^2$ such that

$$\langle a\xi, \xi \rangle + \langle a\eta, \eta \rangle + \sum_{k=1}^m \langle a\zeta_k, \zeta_k \rangle = C \operatorname{tr}(a)$$

for some $C > 0$ and all $a \in \mathbb{M}_2$. The sequence

$$n \mapsto C \operatorname{tr}(a_n) - \langle a_n \xi, \xi \rangle - \langle a_n \eta, \eta \rangle = \sum_{k=1}^m \langle a_n \zeta_k, \zeta_k \rangle$$

is clearly a Stieltjes moment sequence, which shows that the sequence $(\operatorname{tr}(a_n))$ is Stieltjes indeterminate. Hence so is the sequence $(\operatorname{tr}(a_{n+2}))$. Finally note that we have $\operatorname{tr}(a_{n+2}) = 2f(n)$, but we can get $\operatorname{tr}(a_n) = f(n)$, without affecting the other statements of the theorem, by dividing each a_n by 2. $\square$

As far as the growth of positive definite matrix sequences, not of positive type, is concerned, Theorem 2 is not the best possible. Theorem 2 allows us to conclude the existence of a positive definite sequence (a_n) , not of positive type, such that

$$\operatorname{tr}(a_{n+2}) = \delta^{-n^2/2},$$

the right side being a little worse than 2^{n^2} . Something better can be derived from the following result.

Theorem 3. *If $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ is an indeterminate moment sequence, there is a positive definite sequence $(a_n)_{n \in \mathbb{N}_0}$ in $\mathbb{M}_2$, which is not of positive type, such that $\operatorname{tr}(a_{n+2}) = f(n)$ for all $n \in \mathbb{N}_0$.*

Proof. With no restriction, assume $f(0) = 1$. The N -extremal measures σ_t , $t \in \mathbb{R}$, representing f satisfy

$$(11) \quad \int \frac{d\sigma_t(x)}{x-z} = -\frac{A(z)t - C(z)}{B(z)t - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

for certain entire functions A, B, C, D with $A(0) = D(0) = 0$ and $-B(0) = C(0) = 1$ (cf. [1]). Letting z tend to 0 through imaginary values, we get

$$\int \frac{d\sigma_t(x)}{x} = -\frac{1}{t}, \quad t \neq 0.$$

Differentiating (11) with respect to z yields

$$\begin{aligned} \int \frac{d\sigma_t(x)}{(x-z)^2} \\ = -\frac{(A'(z)t - C'(z))(B(z)t - D(z)) - (A(z)t - C(z))(B'(z)t - D'(z))}{(B(z)t - D(z))^2}, \end{aligned}$$

and letting z tend to 0 through imaginary values, we get

$$\int \frac{d\sigma_t(x)}{x^2} = A'(0) - (B'(0) + C'(0))\frac{1}{t} + D'(0)\frac{1}{t^2}.$$

Hence there is some $\beta > 0$ such that

$$\int \frac{d\sigma_{-t}(x)}{x^2} \leq \frac{\beta}{t^2} \quad \text{for } 0 < t \leq 1.$$

Choose $g > 1$, choose $c > \beta g^2$, choose $\varphi > \sqrt{cf(0)}$, define $h > 0$ by $(g + \varphi/h)^2 = c/\beta$, and let $d = 1 + \varphi^2/(g-1)$ and $b = \beta(d + \varphi h)^2$. Now define (a_n) by

$$a_0 = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \quad a_1 = \begin{pmatrix} d & \varphi \\ \varphi & g \end{pmatrix}, \quad a_n = \begin{pmatrix} f(n-2) & 0 \\ 0 & f(n-2) \end{pmatrix}$$

for $n \geq 2$. Suppose $\xi \in \mathbb{C}^2$ with $\|\xi\| = 1$. For $n \geq 2$ we have

$$\langle a_n \xi, \xi \rangle = f(n-2) = \int x^n \frac{d\sigma_{-t}(x)}{x^2}, \quad 0 < t \leq 1.$$

The smallest eigenvalue of a_1 is

$$\frac{d+g}{2} - \sqrt{\left(\frac{d-g}{2}\right)^2 + \varphi^2} = 1.$$

Hence $\langle a_1 \xi, \xi \rangle \geq 1$, and with $t = 1/\langle a_1 \xi, \xi \rangle$ we have

$$\langle a_1 \xi, \xi \rangle = \frac{1}{t} = \int \frac{d\sigma_{-t}(x)}{x} = \int x \frac{d\sigma_{-t}(x)}{x^2}.$$

Finally, with $\xi = (x, y)$, using $2|\operatorname{Re}(x\bar{y})| \leq h|x|^2 + |y|^2/h$,

$$\begin{aligned} & \langle a_0 \xi, \xi \rangle - \beta \langle a_1 \xi, \xi \rangle^2 \\ &= (b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - \beta(d|x|^2 + g|y|^2 + 2\varphi \operatorname{Re}(x\bar{y}))^2 \\ &\geq (b|x|^2 + c|y|^2)(|x|^2 + |y|^2) - \beta((d + \varphi h)|x|^2 + (g + \varphi/h)|y|^2)^2 \\ &= (b - \beta(d + \varphi h)^2)|x|^4 + (c - \beta(g + \varphi/h)^2)|y|^4 \\ &\quad + (b + c - 2\beta(d + \varphi h)(g + \varphi/h))|x|^2|y|^2 \\ &\geq (b - \beta(d + \varphi h)^2)|x|^4 + (c - \beta(g + \varphi/h)^2)|y|^4 \\ &\quad + (b + c - \beta(d + \varphi h)^2 - \beta(g + \varphi/h)^2)|x|^2|y|^2 \\ &= 0, \end{aligned}$$

that is,

$$\langle a_0 \xi, \xi \rangle \geq \frac{\beta}{t^2} \geq \int \frac{d\sigma_{-t}(x)}{x^2}.$$

All of this shows that $(\langle a_n \xi, \xi \rangle)$ is a moment sequence, represented by the measure $x^{-2}\sigma_{-t} + k\varepsilon_0$ for some $k \geq 0$. That (a_n) is not of positive type, follows by a computation like (10). Finally, divide (a_n) by 2 to get $\operatorname{tr}(a_{n+2}) = f(n)$. $\square$

For $c > 1$, the function $f: \mathbb{N}_0 \rightarrow \mathbb{R}$ defined by $f(2n) = \Gamma(c(2n+1))$, $f(2n+1) = 0$, is the moment sequence of the measure with density $\phi(x) = e^{-|x|^{1/c}}/2c$. Since $\int_{-\infty}^{\infty} (\log \phi(x)/(1+x^2))dx > -\infty$, by the criterion of Krein ([1], p. 87) it follows that f is indeterminate. Hence there is a positive definite sequence (a_n) in $\mathbb{M}_2$, not of positive type, such that $\operatorname{tr}(a_{2n+2}) = \Gamma(c(2n+1))$ for all n .

3. CONVOLUTION SEMIGROUPS

If μ is a measure on $\mathbb{R}$ with moments of all orders, we denote by $s\mu: \mathbb{N}_0 \rightarrow \mathbb{R}$ the moment sequence of μ , i.e.,

$$s\mu(n) = \int x^n d\mu(x), \quad n \in \mathbb{N}_0.$$

If μ and ν are measures on $\mathbb{R}$ with moments of all orders, the convolution $\mu * \nu$ again has moments of all orders, and a simple computation shows

$$s(\mu * \nu)(n) = \sum_{k=0}^n \binom{n}{k} s\mu(k) s\nu(n-k).$$

We introduce a binary operation $\circ$ in $\mathbb{R}^{\mathbb{N}_0}$ by

$$\varphi \circ \psi(n) = \sum_{k=0}^n \binom{n}{k} \varphi(k) \psi(n-k),$$

and the preceding formula takes the form

$$s(\mu * \nu) = s\mu \circ s\nu.$$

Note that the indicator $1_{\{0\}}$ is a neutral element for $\circ$. For $\varphi \in \mathbb{R}^{\mathbb{N}_0}$ and $q \in \mathbb{N}_0$ we

write $\varphi^{\circ q} = \overbrace{\varphi \circ \cdots \circ \varphi}^q$ (equal to $1_{\{0\}}$, by definition, in case $q = 0$), and note that

$$\varphi^{\circ q}(n) = n! \sum_{n_1 + \cdots + n_q = n} \prod_{i=1}^q \frac{\varphi(n_i)}{n_i!}.$$

For $\varphi \in \mathbb{R}^{\mathbb{N}_0}$ with $\varphi(0) = 1$ and for $t \in \mathbb{R}_+$ we define $\varphi^{\circ t} \in \mathbb{R}^{\mathbb{N}_0}$ by

$$\varphi^{\circ t}(n) = \sum_{m=0}^{\infty} \binom{t}{m} (\varphi - 1_{\{0\}})^{\circ m}(n), \quad n \in \mathbb{N}_0,$$

noting that the right side has only finitely many terms because of

$$\text{supp}((\varphi - 1_{\{0\}})^{\circ m}) \subset \{m, m+1, \dots\}.$$

This latter fact also shows that $t \mapsto \varphi^{\circ t}$ is continuous with respect to the topology of pointwise convergence on $\mathbb{R}^{\mathbb{N}_0}$. (We always consider $\mathbb{R}^{\mathbb{N}_0}$ with this topology.) If t happens to equal some integer q then $\varphi^{\circ t}$ as newly defined is the same as $\varphi^{\circ q}$ in the previous sense, by the binomial formula.

From the identity $(1+z)^{r+t} = (1+z)^r(1+z)^t$, inserting $(1+z)^{r+t} = \sum_{m=0}^{\infty} \binom{r+t}{m} z^m$ ($|z| < 1$) and similarly for the exponents r and t , we get the identity

$$\binom{r+t}{m} = \sum_{p=0}^m \binom{r}{p} \binom{t}{m-p},$$

and it follows that

$$(12) \quad \varphi^{\circ(r+t)} = \varphi^{\circ r} \circ \varphi^{\circ t}, \quad r, t \in \mathbb{R}_+.$$

For $q \in \mathbb{N}$, the function $\varphi^{\circ 1/q}$ is the unique $\psi \in \mathbb{R}^{\mathbb{N}_0}$ such that $\psi(0) = 1$ and $\psi^{\circ q} = \varphi$. To see this, first note that repeated application of (12) yields $(\varphi^{\circ 1/q})^{\circ q} = \varphi$. Furthermore, suppose $\psi, \omega \in \mathbb{R}^{\mathbb{N}_0}$ are such that $\psi(0) = \omega(0) = 1$ and $\psi^{\circ q} = \omega^{\circ q}$, that is,

$$\sum_{n_1+\dots+n_q=n} \prod_{i=1}^q \frac{\psi(n_i)}{n_i!} = \sum_{n_1+\dots+n_q=n} \prod_{i=1}^q \frac{\omega(n_i)}{n_i!}$$

for $n \in \mathbb{N}_0$. If $n \in \mathbb{N}$ and if $\psi(m) = \omega(m)$ for $m < n$, the terms with $n_i < n$ for all i cancel, and we are left with the terms with $n_i = n$ for one i and $n_i = 0$ for all other i , and the equation simplifies to $\psi(n) = \omega(n)$.

Lemma 1. *If $(\mu_t)_{t \geq 0}$ is a convolution semigroup of probability measures on $\mathbb{R}_+$ with moments of all orders, and if $\varphi = s\mu_1$, then $s\mu_t = \varphi^{\circ t}$ for all $t \in \mathbb{R}_+$.*

Proof. For $q \in \mathbb{N}$ we have $(s\mu_{1/q})^{\circ q} = s\mu_1 = \varphi = (\varphi^{\circ 1/q})^{\circ q}$, hence $s\mu_{1/q} = \varphi^{\circ 1/q}$ by uniqueness. It follows that $s\mu_t = \varphi^{\circ t}$ for all $t \in \mathbb{Q}_+$. Let us show that $(s\mu_t)_{t \geq 0}$ is continuous at 0. Firstly, $s\mu_{1/q} = \varphi^{\circ 1/q} \rightarrow 1_{\{0\}}$ as $\mathbb{N} \ni q \rightarrow \infty$. Secondly, if $m \in \mathbb{N}$, $q \in \mathbb{N}$, and $t \leq 1/q$ then

$$\begin{aligned} s\mu_{1/q}(m) &= s\mu_{1/q-t} \circ s\mu_t(m) = \sum_{k=0}^m \binom{m}{k} s\mu_{1/q-t}(k) s\mu_t(m-k) \\ &\geq s\mu_{1/q-t}(0) s\mu_t(m) = s\mu_t(m), \end{aligned}$$

which shows $s\mu_t \rightarrow 1_{\{0\}}$ as $t \rightarrow 0$. Now for arbitrary $t > 0$ there is a sequence (r_n) of rationals converging to t from the left, and then

$$\begin{aligned} s\mu_t &= \lim_{n \rightarrow \infty} s\mu_{r_n} \circ s\mu_{t-r_n} = \left(\lim_{n \rightarrow \infty} s\mu_{r_n} \right) \circ \left(\lim_{n \rightarrow \infty} s\mu_{t-r_n} \right) \\ &= \left(\lim_{n \rightarrow \infty} \varphi^{\circ r_n} \right) \circ 1_{\{0\}} = \varphi^{\circ t} \end{aligned}$$

since $\circ$ is continuous. □

Definition 2. We denote by ε the number in $]0, 1[$ satisfying $\varepsilon/(1-\varepsilon)^2 = \delta$.

Since $(1/5)/(1 - 1/5)^2 = 5/16 > 1/4 > \delta$ then $\varepsilon < 1/5$. Below we shall find the estimate $\varepsilon > 10/59$. Due to our earlier estimates of δ , one can show $x < \varepsilon < y$ by verifying $x/(1-x)^2 < 0.246315$ and $y/(1-y)^2 > 0.246319$. Using a calculator, one gets $0.169777 < \varepsilon < 0.169780$.

Lemma 2. *If $\varphi: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfies (3) then $\varphi^{\circ t}$ is a Stieltjes indeterminate Stieltjes moment sequence for each $t > 0$.*

Proof. With no loss of generality, assume $\varphi(0) = 1$. For fixed $n \in \mathbb{N}$ we have

$$\varphi^{\circ t}(n) = n! \sum_{m=1}^n \binom{t}{m} S_m$$

where

$$S_m = \sum_{n_1 + \dots + n_m = n, n_i \geq 1 \forall i} \prod_{i=1}^m \frac{\varphi(n_i)}{n_i!}.$$

Now

$$(13) \quad \sum_{k=1}^{n-1} \binom{n}{k} \frac{\varphi(k)\varphi(n-k)}{\varphi(n)} \leq (2^n - 2)\varepsilon^{n-1} \leq 2\varepsilon < 1.$$

To get the first inequality in (13) note that the function $\psi: \mathbb{N}_0 \rightarrow]0, \infty[$ defined by $\psi(j) = \varepsilon^{j^2/2} \varphi(j)$ is log convex according to (3), so for $1 \leq k \leq n-1$ we have $\psi(k)\psi(n-k) \leq \psi(0)\psi(n)$, that is, $\varphi(k)\varphi(n-k)/\varphi(n) \leq \varepsilon^{k(n-k)}$, whence

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{\varphi(k)\varphi(n-k)}{\varphi(n)} \leq \sum_{k=1}^{n-1} \binom{n}{k} \varepsilon^{k(n-k)} \leq (2^n - 2)\varepsilon^{n-1}.$$

In the second inequality in (13) we have strict inequality for $n = 1$ and equality for $n = 2$, and for $n \geq 3$ we have

$$(2^n - 2)\varepsilon^{n-1} \leq 2^n \varepsilon^{n-1} = 8(2\varepsilon)^{n-3} \varepsilon^2 \leq 8\varepsilon^2 < 2\varepsilon$$

since $\varepsilon < 1/4$. This proves (13).

For $m \in \mathbb{N}$, by (13) applied to $n - p$ instead of n ,

$$\begin{aligned} S_{m+1} &= \sum_{p=m-1}^{n-2} \sum_{n_1 + \dots + n_{m-1} = p, n_i \geq 1 \forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!} \right) \sum_{k=1}^{n-p-1} \frac{\varphi(k)\varphi(n-p-k)}{k!(n-p-k)!} \\ &\leq 2\varepsilon \sum_{p=m-1}^{n-2} \sum_{n_1 + \dots + n_{m-1} = p, n_i \geq 1 \forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!} \right) \frac{\varphi(n-p)}{(n-p)!} \\ &\leq 2\varepsilon \sum_{p=m-1}^{n-1} \sum_{n_1 + \dots + n_{m-1} = p, n_i \geq 1 \forall i} \left(\prod_{i=1}^{m-1} \frac{\varphi(n_i)}{n_i!} \right) \frac{\varphi(n-p)}{(n-p)!} \\ &= 2\varepsilon S_m. \end{aligned}$$

For $0 < t < 1$, therefore, the series $\sum_{m=1}^n \binom{t}{m} S_m$ is alternating with absolutely decreasing terms, and so

$$\begin{aligned} (1 - \varepsilon)t\varphi(n) &= n!(1 - \varepsilon)tS_1 \leq n!(1 - (1 - t)\varepsilon)tS_1 \\ &\leq n! \left(tS_1 - \frac{t(1-t)}{2} S_2 \right) \leq \varphi^{\circ t}(n) \leq n!tS_1 = t\varphi(n). \end{aligned}$$

It follows that

$$\frac{\varphi^{\circ t}(n)^2}{\varphi^{\circ t}(n-1)\varphi^{\circ t}(n+1)} \leq \frac{\varphi(n)^2}{(1-\varepsilon)^2\varphi(n-1)\varphi(n+1)} \leq \frac{\varepsilon}{(1-\varepsilon)^2} = \delta,$$

showing that $\varphi^{\circ t}$ is a Stieltjes indeterminate Stieltjes moment sequence for $0 < t < 1$. For $t = 1$ we have the same conclusion directly from (3) since $\varepsilon < \delta$, and for arbitrary $t > 0$ we get the desired conclusion from $\varphi^{\circ t} = \varphi^{\circ r} \circ \varphi^{\circ p}$ where $r \in]0, 1]$ and $p \in \mathbb{N}_0$ are so chosen that $t = r + p$. $\square$

Theorem 4. *If $\varphi: \mathbb{N}_0 \rightarrow]0, \infty[$ satisfies (3), there is a convolution semigroup $(\mu_t)_{t \geq 0}$ of measures on $\mathbb{R}_+$ with moments of all orders such that*

$$\int x^n d\mu_1(x) = \varphi(n), \quad n \in \mathbb{N}_0,$$

and for every such semigroup (μ_t) the measure μ_t is Stieltjes indeterminate for each $t > 0$.

Proof. With no loss of generality, assume $\varphi(0) = 1$. For $t > 0$, by Lemma 2, the function $\varphi^{\circ t}$ is, in particular, a Stieltjes moment sequence, so we can choose a measure λ_t on $\mathbb{R}_+$, with moments of all orders, such that $s\lambda_t = \varphi^{\circ t}$. For $t \rightarrow 0$ we have $s\lambda_t = \varphi^{\circ t} \rightarrow 1_{\{0\}} = s\varepsilon_0$, so $\lambda_t \rightarrow \varepsilon_0$ by the method of moments. Now let (t_i) be a universal subnet of the identical net on $(]0, \infty[, \geq)$. For $t > 0$ we have, denoting by $\lfloor t/t_i \rfloor$ the greatest integer not exceeding t/t_i ,

$$s(\lambda_{t_i}^{*\lfloor t/t_i \rfloor}) = s(\lambda_{t_i})^{\circ \lfloor t/t_i \rfloor} = (\varphi^{\circ t_i})^{\circ \lfloor t/t_i \rfloor} = \varphi^{\circ \lfloor t/t_i \rfloor t_i} \rightarrow \varphi^{\circ t},$$

so by the method of moments, $\lambda_{t_i}^{*\lfloor t/t_i \rfloor} \rightarrow \mu_t$ for some measure μ_t on $\mathbb{R}_+$, with moments of all orders, such that $s\mu_t = \varphi^{\circ t}$. For $r, t > 0$, for each i there is some $m_i \in \{0, 1\}$ such that $\lfloor (r+t)/t_i \rfloor = \lfloor r/t_i \rfloor + \lfloor t/t_i \rfloor + m_i$, and it follows that

$$\mu_{r+t} = \lim \lambda_{t_i}^{*\lfloor (r+t)/t_i \rfloor} = \lim \lambda_{t_i}^{*\lfloor r/t_i \rfloor} * \lambda_{t_i}^{*\lfloor t/t_i \rfloor} * \lambda_{t_i}^{*m_i} = \mu_r * \mu_t.$$

This shows the existence of a convolution semigroup as stated. Now for an arbitrary such semigroup (μ_t) , by Lemma 1 we have $s\mu_t = \varphi^{\circ t}$ for $t > 0$, and the Stieltjes indeterminacy follows from Lemma 2. $\square$

If φ is the moment sequence of the measure (4) with $\sigma > 4/3$ then Theorem 4 applies. To see this, note that $\varphi(n) = e^{\sigma^2 n^2/2}$. Now

$$\sum_{n=1}^{\infty} \left(\frac{15}{61}\right)^{n^2} \leq \sum_{n=1}^{\infty} \left(\frac{15}{61}\right)^{3n-2} = \frac{15 \cdot 61^2}{61^3 - 15^3} = \frac{55815}{223606} < \frac{1}{4},$$

so $\delta > 15/61$. Furthermore,

$$\frac{10/59}{(1-10/59)^2} = \frac{10 \cdot 59}{(59-10)^2} = \frac{590}{2401} < \frac{15}{61} < \delta,$$

so $\varepsilon > 10/59$. But

$$e^{1/9} > 1 + \frac{1}{9} + \frac{1}{2} \left(\frac{1}{9}\right)^2 + \frac{1}{6} \left(\frac{1}{9}\right)^3 = \frac{2444}{2187} > \frac{485}{434},$$

so

$$\begin{aligned} \frac{\varphi(n)^2}{\varphi(n-1)\varphi(n+1)} &= e^{-\sigma^2} < e^{-16/9} \\ &< \left(\frac{434}{485}\right)^{16} < \left(\frac{213}{266}\right)^8 < \left(\frac{84}{131}\right)^4 < \left(\frac{44}{107}\right)^2 < \frac{10}{59} < \varepsilon. \end{aligned}$$

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REFERENCES

- [1] N. I. Akhiezer, *The Classical Moment Problem*, Oliver & Boyd, Edinburgh and London, 1965.
- [2] T. M. Bisgaard, Positive definite operator sequences, *Proc. Amer. Math. Soc.* **121** (1994), 1185–1191. MR **94j**:43004
- [3] T. M. Bisgaard and Z. Sasvári, On the positive definiteness of certain functions, *Math. Nachr.* **186** (1997), 81–99. CMP 97:16
- [4] R. P. Boas, The Stieltjes moment problem for functions of bounded variation, *Bull. Amer. Math. Soc.* **45** (1939), 399–404.

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