

PERIODIC SOLUTIONS FOR NONAUTONOMOUS SECOND ORDER SYSTEMS WITH SUBLINEAR NONLINEARITY

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ABSTRACT. The existence and multiplicity of periodic solutions are obtained for nonautonomous second order systems with sublinear nonlinearity by using the least action principle and the minimax methods.

1. INTRODUCTION AND MAIN RESULTS

Consider the second order systems

$$(1) \quad \begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases}$$

where $T > 0$ and $F: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

Suppose that the nonlinearity $\nabla F(t, x)$ is bounded, that is, there exists $g \in L^1(0, T; \mathbb{R}^+)$ such that

$$|\nabla F(t, x)| \leq g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Mawhin-Willem [3] proved the existence of solutions for problem (1) under the condition that

$$\int_0^T F(t, x) dt \rightarrow +\infty$$

as $|x| \rightarrow \infty$ or that

$$\int_0^T F(t, x) dt \rightarrow -\infty$$

as $|x| \rightarrow \infty$. The recent related results are contained in [1, 2, 3, 5, 6]. In this paper we suppose that the nonlinearity $\nabla F(t, x)$ is sublinear, that is, there exist

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$f, g \in L^1(0, T; \mathbb{R}^+)$ and $a \in [0, 1[$ such that

$$(2) \quad |\nabla F(t, x)| \leq f(t)|x|^\alpha + g(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Then the existence of periodic solutions, which generalizes Mawhin-Willem's results mentioned above, are obtained for the nonautonomous second order systems with sublinear nonlinearity by using the least action principle and the minimax methods. Moreover the multiplicity of periodic solutions is also obtained. The main results are the following theorems.

Theorem 1. *Suppose that (2) and assumption (A) hold. Assume that*

$$(3) \quad |x|^{-2\alpha} \int_0^T F(t, x) dt \rightarrow +\infty$$

as $|x| \rightarrow \infty$. Then problem (1) has at least one solution which minimizes the function φ given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, 0)] dt$$

on the Hilbert space H_T^1 defined by

$$H_T^1 = \{u : [0, T] \rightarrow \mathbb{R}^N \mid u \text{ is absolutely continuous,} \\ u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N)\}$$

with the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}, \quad u \in H_T^1.$$

Remark 1. Theorem 1 generalizes Theorem 1.5 of Mawhin-Willem [3]. In fact, it follows from Theorem 1 by letting $\alpha = 0$. On the other hand, there are functions F satisfying our Theorem 1 and not satisfying the results in [1, 2, 3, 5, 6]. For example, let $\alpha = \frac{1}{2}$ and

$$F(t, x) = \left(\frac{2}{3}T - t \right) |x|^{\frac{3}{2}} + (h(t), x)$$

where $h \in L^1(0, T; \mathbb{R}^N)$.

Theorem 2. *Suppose that (2) and assumption (A) hold. Assume that*

$$(4) \quad |x|^{-2\alpha} \int_0^T F(t, x) dt \rightarrow -\infty$$

as $|x| \rightarrow \infty$. Then problem (1) has at least one solution in H_T^1 .

Remark 2. Theorem 2 generalizes Theorem 4.8 of Mawhin-Willem [3]. In fact, it follows from Theorem 2 by letting $\alpha = 0$. There are functions F satisfying our Theorem 2 and not satisfying the results in [1, 2, 3, 5, 6]. For example, let $\alpha = \frac{1}{2}$ and

$$F(t, x) = \left(\frac{1}{3}T - t \right) |x|^{\frac{3}{2}} + (h(t), x)$$

where $h \in L^1(0, T; \mathbb{R}^N)$.

Theorem 3. Suppose that (2), (4) and assumption (A) hold. Assume that there exist $\delta > 0, \varepsilon > 0$ and an integer $k > 0$ such that

$$(5) \quad -\frac{1}{2}(k+1)^2\omega^2|x| \leq F(t, x) - F(t, 0)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and

$$(6) \quad F(t, x) - F(t, 0) \leq -\frac{1}{2}k^2\omega^2(1+\varepsilon)|x|^2$$

for all $|x| \leq \delta$ and a.e. $t \in [0, T]$, where $\omega = \frac{2\pi}{T}$. Then problem (1) has at least two distinct solutions in H_T^1 .

Theorem 4. Suppose that (2), (3) and assumption (A) hold. Assume that there exist $\delta > 0$ and an integer $k \geq 0$ such that

$$(7) \quad -\frac{1}{2}(k+1)^2\omega^2|x|^2 \leq F(t, x) - F(t, 0) \leq -\frac{1}{2}k^2\omega^2|x|^2$$

for all $|x| \leq \delta$ and a.e. $t \in [0, T]$. Then problem (1) has at least three distinct solutions in H_T^1 .

Remark 3. There are functions F satisfying our Theorem 4 and not satisfying Theorem 7 in [1] and its generalization in [6]. For example, let

$$F(t, x) = \begin{cases} \left(\frac{2}{3}T - t\right)|x|^{\frac{3}{2}}, & |x| \geq 1, \\ -\frac{1}{4}\omega^2|x|^2 + \left(\frac{1}{2}\omega^2 + \frac{3}{2}T - \frac{9}{4}t\right)|x|^4 - \left(\frac{1}{4}\omega^2 + \frac{5}{6}T - \frac{5}{4}t\right)|x|^6, & |x| \leq 1, \end{cases}$$

for all $t \in [0, T]$.

2. PROOFS OF THE THEOREMS

For $u \in H_T^1$, let $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$. Then one has

$$\|\tilde{u}\|_\infty^2 \leq \frac{T}{12} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Sobolev's inequality})$$

and

$$\int_0^T |\tilde{u}(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{u}(t)|^2 dt \quad (\text{Wirtinger's inequality}).$$

It follows from assumption (A) that the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, 0)] dt$$

is continuously differentiable and weakly lower semicontinuous on H_T^1 (see [3]). Moreover one has

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t)) dt + \int_0^T (\nabla F(t, u(t)), v(t)) dt$$

for all $u, v \in H_T^1$. It is well-known that the solutions of problem (1) correspond to the critical points of φ .

Proof of Theorem 1. It follows from (2) and Sobolev's inequality that

$$\begin{aligned}
 & \left| \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt \right| \\
 &= \left| \int_0^T \int_0^1 (\nabla F(t, \bar{u} + s\tilde{u}(t)), \tilde{u}(t)) ds dt \right| \\
 &\leq \int_0^T \int_0^1 f(t) |\bar{u} + s\tilde{u}(t)|^\alpha |\tilde{u}(t)| ds dt + \int_0^T \int_0^1 g(t) |\tilde{u}(t)| ds dt \\
 &\leq 2(|\bar{u}|^\alpha + \|\tilde{u}\|_\infty^\alpha) \|\tilde{u}\|_\infty \int_0^T f(t) dt + \|\tilde{u}\|_\infty \int_0^T g(t) dt \\
 &\leq \frac{3}{T} \|\tilde{u}\|_\infty^2 + \frac{T}{3} |\bar{u}|^{2\alpha} \left(\int_0^T f(t) dt \right)^2 + 2\|\tilde{u}\|_\infty^{\alpha+1} \int_0^T f(t) dt + \|\tilde{u}\|_\infty \int_0^T g(t) dt \\
 &\leq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt + C_1 |\bar{u}|^{2\alpha} + C_2 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} + C_3 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}
 \end{aligned}$$

for all $u \in H_T^1$. Hence we have

$$\begin{aligned}
 \varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, \bar{u})] dt + \int_0^T F(t, \bar{u}) dt - \int_0^T F(t, 0) dt \\
 &\geq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt - C_2 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} - C_3 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}} \\
 &\quad + |\bar{u}|^{2\alpha} \left(|\bar{u}|^{-2\alpha} \int_0^T F(t, \bar{u}) dt - C_1 \right) - \int_0^T F(t, 0) dt
 \end{aligned}$$

for all $u \in H_T^1$. As $\|u\| \rightarrow \infty$ if and only if $(|\bar{u}|^2 + \int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \rightarrow \infty$, the above equality and (3) imply that

$$\varphi(u) \rightarrow +\infty$$

as $\|u\| \rightarrow \infty$. By Theorem 1.1 and Corollary 1.1 in [3] we complete our proof.

Proof of Theorem 2. First we prove that φ satisfies the (PS) condition. Suppose that (u_n) is a (PS) sequence for φ , that is, $\varphi'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ and $\{\varphi(u_n)\}$ is bounded. In a way similar to the proof of Theorem 1, we have

$$\begin{aligned}
 \left| \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \right| &\leq \frac{1}{4} \int_0^T |\dot{u}_n(t)|^2 dt + C_1 |\bar{u}_n|^{2\alpha} \\
 &\quad + C_2 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}} + C_3 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}
 \end{aligned}$$

for all n . Hence one has

$$\begin{aligned}\|\tilde{u}_n\| &\geq \langle \varphi'(u_n), \tilde{u}_n \rangle \\ &= \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T (\nabla F(t, u_n(t)), \tilde{u}_n(t)) dt \\ &\geq \frac{3}{4} \int_0^T |\dot{u}_n(t)|^2 dt - C_1 |\bar{u}_n|^{2\alpha} \\ &\quad - C_2 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}} - C_3 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}\end{aligned}$$

for large n . It follows from Wirtinger's inequality that

$$\|\tilde{u}_n\| \leq \left(\frac{T^2}{4\pi^2} + 1 \right)^{\frac{1}{2}} \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}$$

for all n . Hence we obtain

$$(8) \quad C |\bar{u}_n|^\alpha \geq \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}$$

for some $C > 0$ and all large n . By the proof of Theorem 1 we have

$$\begin{aligned}\int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)] dt &\leq \frac{1}{4} \int_0^T |\dot{u}_n(t)|^2 dt + C_1 |\bar{u}_n|^{2\alpha} \\ &\quad + C_2 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}} + C_3 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}}\end{aligned}$$

for all n . It follows from the boundedness of $\{\varphi(u_n)\}$, (8) and the above inequality that

$$\begin{aligned}C_4 &\leq \varphi(u_n) \\ &= \frac{1}{2} \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T [F(t, u_n(t)) - F(t, \bar{u}_n)] dt \\ &\quad + \int_0^T F(t, \bar{u}_n) dt - \int_0^T F(t, 0) dt \\ &\leq \frac{3}{4} \int_0^T |\dot{u}_n(t)|^2 dt + C_1 |\bar{u}_n|^{2\alpha} + C_2 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{\alpha+1}{2}} \\ &\quad + C_3 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} + \int_0^T F(t, \bar{u}_n) dt - \int_0^T F(t, 0) dt \\ &\leq |\bar{u}_n|^{2\alpha} \left(|\bar{u}_n|^{-2\alpha} \int_0^T F(t, \bar{u}_n) dt + C_5 \right) - \int_0^T F(t, 0) dt\end{aligned}$$

for all large n and some real constant C_4 and C_5 . The above inequality and (4) implies that $(|\bar{u}_n|)$ is bounded. Hence (u_n) is bounded by (8). Arguing then as in Proposition 4.1 in [3], we conclude that the (PS) condition is satisfied.

Let \tilde{H}_T^1 be the subspace of H_T^1 given by

$$\tilde{H}_T^1 = \{u \in H_T^1 | \bar{u} = 0\}.$$

Then we have

$$(9) \quad \varphi(u) \rightarrow +\infty$$

as $\|u\| \rightarrow \infty$ in \tilde{H}_T^1 . In fact, by the proof of Theorem 1 one has

$$\begin{aligned} & \left| \int_0^T [F(t, u(t)) - F(t, 0)] dt \right| \\ & \leq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt + C_2 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} + C_3 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

for all $u \in \tilde{H}_T^1$. It follows that

$$\begin{aligned} \varphi(u) &= \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T [F(t, u(t)) - F(t, 0)] dt \\ &\geq \frac{1}{4} \int_0^T |\dot{u}(t)|^2 dt - C_2 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{\alpha+1}{2}} - C_3 \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

for all $u \in \tilde{H}_T^1$. By Wirtinger's inequality, the norm

$$|||u||| = \left(\int_0^T |\dot{u}(t)|^2 dt \right)^{\frac{1}{2}}$$

is an equivalent norm on \tilde{H}_T^1 . Hence (9) follows from the equivalence and the above inequality.

On the other hand, one has

$$(10) \quad \varphi(u) \rightarrow -\infty$$

as $|u| \rightarrow \infty$ in R^N , which follows from (4). Now Theorem 2 is proved by (9), (10) and the Saddle Point Theorem (see Theorem 4.6 in [4]).

Proof of Theorem 3. Let $E = H_T^1$,

$$(11) \quad H_k = \left\{ \sum_{j=0}^k (a_j \cos j\omega t + b_j \sin j\omega t) | a_j, b_j \in R^N, j = 0, \dots, k \right\}$$

and $\psi = -\varphi$. Then $\psi \in C^1(E, R)$ satisfies the (PS) condition. By Theorem 5.29 and Example 5.26 in [4], we only need to prove

(ψ_1) $\liminf \|u\|^{-2} \psi(u) > 0$ as $u \rightarrow 0$ in H_k ,

(ψ_2) $\psi(u) \leq 0$ for all $u \in H_k^\perp$, and

(ψ_3) $\psi(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$ in H_{k-1}^\perp .

Notice that

$$F(t, x) - F(t, 0) = \int_0^1 (\nabla F(t, sx), x) ds$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. By (2) we have

$$F(t, x) - F(t, 0) \leq \frac{f(t)}{1+\alpha} |x|^{\alpha+1} + g(t)|x| \leq h(t)|x|^3$$

for all $|x| \geq \delta$, a.e. $t \in [0, T]$ and some $h \in L^1(0, T; R^+)$ given by

$$h(t) = \frac{\delta^{\alpha-2}}{1+\alpha} f(t) + \delta^{-2} g(t).$$

Now it follows from (6) that

$$F(t, x) - F(t, 0) \leq -\frac{1}{2} k^2 \omega^2 (1 + \varepsilon) |x|^2 + h(t) |x|^3$$

for all $x \in R^N$ and a.e. $t \in [0, T]$. Hence we obtain

$$\begin{aligned} \psi(u) &\geq -\frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} k^2 \omega^2 (1 + \varepsilon) \int_0^T |u(t)|^2 dt - \int_0^T h(t) |u(t)|^3 dt \\ (12) \quad &\geq \frac{1}{2} \varepsilon \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} k^2 \omega^2 (1 + \varepsilon) |\bar{u}|^2 T - \|u\|_\infty^3 \int_0^T h(t) dt \\ &\geq C_6 \|u\|^2 - C_7 \|u\|^3 \end{aligned}$$

for all $u \in H_k$, where $C_6 = \min\{\frac{\varepsilon}{2}, \frac{1}{2} k^2 \omega^2 (1 + \varepsilon) T\}$, $C_7 = C^3 \int_0^T h(t) dt$ and C is a positive constant such that

$$(13) \quad \|u\|_\infty \leq C \|u\|$$

for all $u \in H_T^1$ (see Proposition 1.1 in [3]). Now (ψ_1) follows from (12). For $u \in H_k^\perp$, one has

$$\psi(u) \leq -\frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \frac{1}{2} (k+1)^2 \omega^2 \int_0^T |u(t)|^2 dt \leq 0$$

which is (ψ_2) . At last (ψ_3) follows from (9). Hence the proof of Theorem 3 is completed.

Proof of Theorem 4. From the proof of Theorem 1 we know that φ is coercive, which implies that φ satisfies the (PS) condition. Let X_2 be the finite-dimensional subspace H_k given by (11) and let $X_1 = X_2^\perp$. Then by (7) we have

$$\varphi(u) \leq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} k^2 \omega^2 \int_0^T |u(t)|^2 dt \leq 0$$

for all $u \in X_2$ with $\|u\| \leq C^{-1} \delta$ and

$$\varphi(u) \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2} (k+1)^2 \omega^2 \int_0^T |u(t)|^2 dt \geq 0$$

for all $u \in X_1$ with $\|u\| \leq C^{-1} \delta$, where C is the positive constant given in (13).

The case that $\int_0^T [F(t, x) - F(t, 0)] dt < 0$ for some $|x| < \delta$, implies $\inf \varphi < 0$. Now our Theorem 4 follows from Theorem 4 in [1].

On the contrary we have $\int_0^T [F(t, x) - F(t, 0)] dt \geq 0$ for all $|x| < \delta$. Then it follows from (7) that for every given $|x| < \delta$ one has $F(t, x) - F(t, 0) = 0$ for a.e. $t \in [0, T]$. Let

$$E(x) = \{t \in [0, T] | F(t, x) - F(t, 0) \neq 0\}.$$

Then $\text{meas } E(x) = 0$ for all $|x| < \delta$. Given $|x_0| < \delta$ we have

$$\left| x_0 + \frac{1}{n} e_m \right| < \delta$$

for $n > \frac{1}{\delta - |x_0|}$, where $\{e_m | 1 \leq m \leq N\}$ is the canonical basis of R^N . Thus we obtain

$$(\nabla F(t, x_0), e_m) = \lim_{n \rightarrow \infty} \frac{F(t, x_0 + \frac{1}{n}e_m) - F(t, x_0)}{\frac{1}{n}} = 0$$

for all $t \notin (\bigcup\{E(x_0 + \frac{1}{n}e_m) | n > \frac{1}{\delta - |x_0|}, 1 \leq m \leq N\}) \cup E(x_0)$, which implies that $\nabla F(t, x_0) = 0$ for a.e. $t \in [0, T]$, that is, x_0 is a solution of problem (1). Hence all $|x| < \delta$ are solutions of problem (1). Therefore Theorem 4 is proved.

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