A CHARACTERIZATION OF UNIFORM CONTINUITY FOR VOLTERRA EQUATIONS IN HILBERT SPACES

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ABSTRACT. We show that the norm continuity of the resolvent for a Volterra equation of scalar type is equivalent to the decay to zero of a holomorphic operator family along some imaginary axis.

1. INTRODUCTION

Let X be a complex Banach space. In this work we consider the following Volterra equation of scalar type

$$(1.1) \quad u(t) = \int_0^t a(t-s)Au(s)ds + f(t), \quad t \in J := [0, T],$$

where A is a closed linear, densely defined operator in X , and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel. A continuous function $u : J \rightarrow X$ is called a strong solution of (1.1) on J if $u(t) \in D(A)$ for all $t \in J$, $Au(t)$ is continuous, and (1.1) holds on J . The basic concept concerning (1.1) is that of well-posedness which is the direct extension of the corresponding notion usually employed for the abstract Cauchy problem (of first order)

$$(1.2) \quad \dot{u}(t) = Au(t), \quad u(0) = u_0.$$

In [4] it is shown that well-posedness is equivalent to the existence of a resolvent $\{S(t)\}_{t \geq 0} \subseteq \mathcal{B}(X)$ for (1.1), i.e. a strongly continuous family of bounded linear operators in X which commutes with A and satisfies the resolvent equation

$$(1.3) \quad S(t)x = x + \int_0^t a(t-s)AS(s)xds, \quad t \geq 0, \quad x \in D(A).$$

The resolvent is the central object to be studied in the theory of Volterra equations; it corresponds to the strongly continuous semigroup generated by A in the special case $a(t) = 1$, i.e. for (1.2). The importance of the resolvent $S(t)$ is shown

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by the variation of parameters formula

$$(1.4) \quad u(t) = S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)ds, \quad t \in J,$$

where $f \in W^{1,1}(J; X)$.

Due to the time invariance of (1.1), Laplace transform methods can be employed. Formally the Laplace transform $H(\lambda) = \hat{S}(\lambda)$ of the resolvent is represented by

$$(1.5) \quad H(\lambda) = (\lambda - \lambda \hat{a}(\lambda)A)^{-1}.$$

This formula leads to a characterization of Hille-Yosida type for existence of an at most exponentially growing resolvent for (1.1) in terms of properties of the holomorphic operator family $H(\lambda)$. For a general exposition of the theory, we refer to J. Prüss [4].

In this article we characterize the uniform continuity of the resolvent $S(t)$ for (1.1) on Hilbert spaces in terms of (1.5). We show that the norm continuity of $S(t)$ for $t > 0$ is equivalent to the decay to zero of $H(\lambda)$ along some imaginary axis. For the special case $a(t) = 1$ and A generator of a C_0 -semigroup $T(t)$ in H , we have $S(t) = T(t)$ and the result reduces to a characterization of eventually norm continuous semigroups obtained recently by P. You [5] (see also [1] and [3]).

2. A CHARACTERIZATION IN HILBERT SPACES

Let H be a complex Hilbert space, A a closed linear unbounded operator in H with dense domain $D(A)$, and $a \in L^1_{loc}(\mathbb{R}_+)$ a scalar kernel $\neq 0$. We consider the Volterra equation

$$(2.1) \quad u(t) = f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in J,$$

where $f \in C(J; H)$, $J := [0, T]$.

Since (2.1) is a convolution equation on the halfline it is natural to employ the Laplace transform for its study. Besides the standing assumptions on $a(t)$ and A we therefore suppose that $a(t)$ is Laplace transformable, i.e. there is $\omega \in \mathbb{R}$ such that $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. But we also have to restrict the class of resolvents.

We recall that $S(t)$ is called exponentially bounded of type (M, w) , if there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \text{for all } t \geq 0.$$

We will also need the following definition from [4].

Definition 2.1. Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and $k \in \mathbb{N}$. $a(t)$ is called k -regular if there is a constant $C > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq C|\hat{a}(\lambda)|$$

for all $\operatorname{Re} \lambda > \omega$, $0 < n \leq k$.

Convolutions of k -regular kernels are again k -regular. Moreover, integration and differentiation are operations which preserve k -regularity as well. See [4], p. 70.

Our main result in this article is the following characterization.

Theorem 2.2. Let A be a closed linear unbounded operator in a Hilbert space H with dense domain $D(A)$ and let $a \in L^1_{loc}(\mathbb{R}_+)$ satisfy $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$ for some $\omega \in \mathbb{R}$. Assume (2.1) admits a resolvent $S(t)$ of type (M, w) , and $a(t)$ is 2-regular. Then the following conditions are equivalent.

- (a) $(S(t))_{t \geq 0}$ is continuous in $\mathcal{B}(H)$ for $t > 0$,
 (b) $\lim_{|\mu| \rightarrow \infty} \|H(\mu_0 + i\mu)\| = 0$ for some $\mu_0 > \omega$.

The above theorem reduces for the case $a(t) \equiv 1$ to the following result of P. You [5].

Corollary 2.3. *Let A be the generator of a strongly continuous semigroup $T(t)$ on a Hilbert space H . Then the following conditions are equivalent.*

- (a) $(T(t))_{t \geq 0}$ is norm continuous for $t > 0$,
 (b) $\lim_{|\mu| \rightarrow \infty} \|R(\mu_1 + i\mu; A)\| = 0$ for some $\mu_1 > \omega$.

Uniform continuity of the resolvent $S(t)$ has a number of important consequences. For example, $u(t)$ defined by (1.4) in general is a solution of (1.1) if $Af \in W^{1,1}(J; H)$ or $f = a * g$ with $g \in W^{1,1}(J; H)$; if $S(t)$ is $\mathcal{B}(H)$ -continuous, then $Af \in BV(J; H)$ or $f = a * g$ with $g \in BV(J; H)$ are already sufficient.

Another type of application concerns stability of (1.1) or, in other terms, integrability of $S(t)$ (see [4], section 10).

For example, suppose $b \in L^1_{loc}(\mathbb{R}_+)$ satisfies $\int_0^\infty |b(t)|e^{-\beta t} dt < \infty$ for some $\beta \geq 0$, and assume there is a resolvent $S_b(t)$ for

$$v(t) = g(t) + \int_0^t b(t-s)Av(s)ds, \quad t \in \mathbb{R}_+,$$

such that $\omega_b = \overline{\lim}_{t \rightarrow \infty} t^{-1} \log |S_b(t)| < \infty$.

Let $c(t)$ be a completely positive function with associated creep function

$$k(t) = \kappa + \omega t + \int_0^t k_1(s)ds, \quad t > 0,$$

where $\kappa, \omega \geq 0, k_1(t) \geq 0$ nonincreasing with $\lim_{t \rightarrow \infty} k_1(t) = 0$, and let $\alpha = \omega + k_1(0^+)$. According to [4], Theorem 4.1, there is a resolvent $S_a(t)$ for (1.1), where $a \in L^1_{loc}(\mathbb{R}_+)$ is defined by $\hat{a}(\lambda) = \hat{b}(1/\hat{c}(\lambda))$, for $\operatorname{Re} \lambda$ sufficiently large.

As a direct consequence of Theorem 2.2 and [4], Theorem 10.4, we obtain the following result.

Theorem 2.4. *Let H be a Hilbert space. Suppose the function*

$$g(z, \omega) = w^{-1}(\hat{b}(z)/\hat{b}(z + z\omega) - 1)$$

is analytic at $(\infty, 0)$, let $1/\kappa + \alpha < \infty$ and assume $\alpha > \omega_b$, and $b(t)$ is 2-regular. Then $S_a(t)$ is uniformly integrable if $S_a(t)$ is integrable and

$$\lim_{|\mu| \rightarrow \infty} \|H_b(\mu_0 + i\mu)\| = 0$$

for some sufficiently large μ_0 .

In order to prove Theorem 2.2 we will need the following lemmas.

Lemma 2.5. *Let $a \in L^1_{loc}(\mathbb{R}_+)$ be Laplace transformable and assume that*

$$H(\lambda) = (\lambda - \lambda \hat{a}(\lambda)A)^{-1}$$

exists for all $\operatorname{Re} \lambda > \omega$. Then there are functions $k_1(\lambda), k_2(\lambda)$ and $h_1(\lambda), h_2(\lambda), h_3(\lambda)$ such that

- (i) $H'(\lambda) = k_1(\lambda)H(\lambda) + k_2(\lambda)H(\lambda)^2$, $\operatorname{Re} \lambda > \omega$,
 (ii) $H''(\lambda) = h_1(\lambda)H(\lambda) + h_2(\lambda)H(\lambda)^2 + h_3(\lambda)H(\lambda)^3$, $\operatorname{Re} \lambda > \omega$.

Proof. First, we observe that $\hat{a}(\lambda) \neq 0$ for $\operatorname{Re} \lambda > \omega$ because A is unbounded ([4], p. 43). Next, let $H(\lambda) = p(\lambda)(q(\lambda) - A)^{-1}$ where $p(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)}$ and $q(\lambda) := \frac{1}{\hat{a}(\lambda)}$. Then,

$$H'(\lambda) = \frac{p'(\lambda)}{p(\lambda)}H(\lambda) - \frac{q'(\lambda)}{p(\lambda)}H(\lambda)^2, \quad \operatorname{Re} \lambda > \omega,$$

and

$$H''(\lambda) = \frac{p''(\lambda)}{p(\lambda)}H(\lambda) + \left[\frac{2q'(\lambda)p'(\lambda)}{p(\lambda)^2} - \frac{q''(\lambda)}{p(\lambda)} \right] H(\lambda)^2 - \frac{2q'(\lambda)^2}{p(\lambda)^2} H(\lambda)^3, \quad \operatorname{Re} \lambda > \omega.$$

□

Remark. A calculation shows us that

$$(2.2) \quad k_1(\lambda) := -\left(\frac{1}{\lambda} + \frac{\hat{a}'(\lambda)}{\hat{a}(\lambda)}\right), \quad k_2(\lambda) := \frac{\lambda \hat{a}'(\lambda)}{\hat{a}(\lambda)}, \quad \operatorname{Re} \lambda > \omega,$$

and

$$(2.3) \quad h_1(\lambda) := \frac{2\hat{a}'(\lambda)}{\lambda \hat{a}(\lambda)} + \frac{2}{\lambda^2} + \frac{2\hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2} - \frac{\hat{a}''(\lambda)}{\hat{a}(\lambda)}, \quad \operatorname{Re} \lambda > \omega,$$

$$(2.4) \quad h_2(\lambda) := \frac{2\hat{a}'(\lambda)}{\hat{a}(\lambda)} + \frac{\lambda \hat{a}''(\lambda)}{\hat{a}(\lambda)}, \quad h_3(\lambda) := \frac{-2\lambda^2 \hat{a}'(\lambda)^2}{\hat{a}(\lambda)^2}, \quad \operatorname{Re} \lambda > \omega.$$

As a direct consequence of formulae (2.2)-(2.4) we obtain the following:

Lemma 2.6. *Let $a \in L_{loc}^1(\mathbb{R}_+)$ be Laplace transformable and 2-regular, then there exists a constant $M > 0$ such that*

- (i) $|\lambda^{1-j} k_{1+j}(\lambda)| < M$ for all $\operatorname{Re} \lambda > \omega$; $j = 0, 1$,
- (ii) $\int_{|\mu| \geq N} |h_j(\mu_0 + i\mu)|^j d\mu < M$ for all $\mu_0 > \omega$ and $N > 1$; $j = 1, 2$,
- (iii) $\sup_{|\mu| \geq N} |h_3(\mu_0 + i\mu)| < M$ for all $\mu_0 > \omega$ and $N > 1$.

We will need also the following result from [1].

Lemma 2.7. *Let X be a Banach space and let $R : [0, \infty) \rightarrow X$ be a function which is continuous for $t > 0$. Moreover, assume that there exist $M > 0, \omega \in \mathbb{R}$ such that $\|R(t)\| \leq Me^{\omega t}$. Then the Laplace transform $\hat{R}(\lambda) := \int_0^\infty e^{-\lambda t} R(t) dt$ exists for all $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > \omega$ and*

$$\lim_{|\mu| \rightarrow \infty} \hat{R}(\mu_0 + i\mu) = 0$$

for every $\mu_0 > \omega$.

We can now prove our main result. The proof is inspired by the proof of O. El Mennaoui and K.-J. Engel [1] for the semigroup case.

Proof of Theorem 2.2. (a) implies (b). We use Lemma 2.7 choosing $X = \mathcal{B}(H)$ and $R(t) = S(t)$.

(b) implies (a). Let $x \in H$ be fixed. Because $\|S(t)e^{-\mu_0 t} x\| \leq Me^{\omega_0 t} \|x\|$ with $\omega_0 := \omega - \mu_0 < 0$, the function $t \rightarrow \chi_{[0, \infty)}(t) e^{-\mu_0 t} S(t)x$ is in $L^2(\mathbb{R}; H)$, where $\chi_{[0, \infty)}(t)$ denotes the characteristic.

Since H is a Hilbert space, by Plancherel's theorem the Fourier transform is a unitary operator on $L^2(\mathbb{R}; H)$. Thus we obtain

$$\mathcal{F}(\chi_{[0,\infty)}(\cdot)e^{-\mu_0\cdot}S(\cdot)) = H(\mu_0 + i\mu)x.$$

Hence,

$$(2.5) \quad S(t)e^{-\mu_0 t}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t} H(\mu_0 + i\mu)x d\mu,$$

for $t > 0$ and each $x \in H$. Clearly, the resolvent $S(t)$ is continuous in $\mathcal{B}(H)$ for $t > 0$ if and only if $S_{\mu_0}(t) := S(t)e^{-\mu_0 t}$ is continuous in $\mathcal{B}(H)$ for $t > 0$. Next, note that for $x \in D(A)$ we have

$$H(\mu_0 + i\mu)x = \hat{a}(\mu_0 + i\mu)H(\mu_0 + i\mu)Ax + \frac{1}{\mu_0 + i\mu}x.$$

Observe that $\hat{a}(\mu_0 + i\mu) \rightarrow 0$ as $|\mu| \rightarrow \infty$ by the Riemann-Lebesgue lemma. In particular $\lim_{|\mu| \rightarrow \infty} H(\mu_0 + i\mu)x = 0$. Using this and integrating by parts in (2.5) we obtain

$$S_{\mu_0}(t)x = \frac{-1}{2\pi t} \int_{-\infty}^{\infty} e^{i\mu t} H'(\mu_0 + i\mu)x d\mu$$

for all $x \in D(A)$, $t > 0$. Next, by Lemma 2.5(i) and Lemma 2.6(i) we also get $\lim_{|\mu| \rightarrow \infty} H'(\mu_0 + i\mu)x = 0$. Using this and again integration by parts we obtain the formula

$$(2.6) \quad S_{\mu_0}(t)x = \frac{1}{\pi t^2} \int_{-\infty}^{\infty} e^{i\mu t} H''(\mu_0 + i\mu)x d\mu$$

for all $x \in D(A)$, $t > 0$.

Next we show that the operator family $(t^2 S_{\mu_0}(t))$ is continuous in $\mathcal{B}(H)$ for $t > 0$. In fact, formula (2.6) shows

$$\begin{aligned} \|t^2 S_{\mu_0}(t)x - s^2 S_{\mu_0}(s)x\| &= \frac{1}{\pi} \left\| \int_{-\infty}^{\infty} (e^{i\mu t} - e^{i\mu s}) H''(\mu_0 + i\mu)x d\mu \right\| \\ &\leq \frac{1}{\pi} \left\| \int_{|\mu| \geq N} (e^{i\mu t} - e^{i\mu s}) H''(\mu_0 + i\mu)x d\mu \right\| \\ &\quad + \frac{1}{\pi} \int_{|\mu| \leq N} |e^{i\mu t} - e^{i\mu s}| \|H''(\mu_0 + i\mu)x\| d\mu \\ &=: I_1(N) + I_2(N). \end{aligned}$$

Let $\epsilon > 0$. We show that $I_1(N) < \epsilon \|x\|$ for N sufficiently large. To this end take $x^* \in H$ and observe that by Lemma 2.5(ii) we have

$$\begin{aligned} & \frac{1}{2} \left| \left\langle \int_{|\mu| \geq N} (e^{i\mu t} - e^{i\mu s}) H''(\mu_0 + i\mu) x d\mu, x^* \right\rangle \right| \\ & \leq \int_{|\mu| \geq N} |\langle h_1(\mu_0 + i\mu) H(\mu_0 + i\mu) x, x^* \rangle| d\mu \\ & \quad + \int_{|\mu| \geq N} |\langle h_2(\mu_0 + i\mu) H(\mu_0 + i\mu)^2 x, x^* \rangle| d\mu \\ & \quad + \int_{|\mu| \geq N} |\langle h_3(\mu_0 + i\mu) H(\mu_0 + i\mu)^3 x, x^* \rangle| d\mu \\ & \leq \text{Sup}_{|\mu| \geq N} \|H(\mu_0 + i\mu)\| \left(\left(\int_{|\mu| \geq N} |h_1(\mu_0 + i\mu)| d\mu \right) \|x\| \|x^*\| \right. \\ & \quad + \left(\int_{|\mu| \geq N} \|H(\mu_0 + i\mu)x\|^2 d\mu \right)^{1/2} \left(\int_{|\mu| \geq N} \|h_2(\mu_0 + i\mu)x^*\|^2 d\mu \right)^{1/2} \\ & \quad + \text{Sup}_{|\mu| \geq N} |h_3(\mu_0 + i\mu)| \left(\int_{|\mu| \geq N} \|H(\mu_0 + i\mu)x\|^2 d\mu \right)^{1/2} \\ & \quad \left. \times \left(\int_{|\mu| \geq N} \|H(\mu_0 + i\mu)^* x^*\|^2 d\mu \right)^{1/2} \right) \end{aligned}$$

where we used the Cauchy-Schwartz and Hölder inequalities. Next, by the Plancherel theorem for the Hilbert space valued Fourier transform (see [2], Lemma 2) and Lemma 2.6(ii) and (iii), we get the estimate

$$\begin{aligned} & \leq \text{Sup}_{|\mu| \geq N} \|H(\mu_0 + i\mu)\| \left(M \|x\| \|x^*\| + (2\pi M \int_0^\infty \|S_{\mu_0}(t)x\|^2 dt)^{1/2} \|x^*\| \right. \\ & \quad + \left. M (2\pi \int_0^\infty \|S_{\mu_0}(t)x\|^2 dt)^{1/2} (2\pi \int_0^\infty \|S_{\mu_0}(t)^* x^*\|^2 dt)^{1/2} \right). \end{aligned}$$

Since $(S_{\mu_0}(t))_{t \geq 0}$ and the adjoint $(S_{\mu_0}(t)^*)_{t \geq 0}$ are exponentially bounded of type (M, ω_0) for some $\omega_0 < 0$, there exists a constant $C > 0$ such that

$$2\pi \int_0^\infty \|S_{\mu_0}(t)x\|^2 dt \leq C^2 \|x\| \quad \text{and} \quad 2\pi \int_0^\infty \|S_{\mu_0}(t)^* x^*\|^2 dt \leq C^2 \|x^*\|.$$

Combining this with the above estimate we obtain a constant $K > 0$ such that

$$\begin{aligned} I_1(N) &= \text{Sup}_{\|x^*\| \leq 1} \left| \left\langle \int_{|\mu| \geq N} (e^{i\mu t} - e^{i\mu s}) H''(\mu_0 + i\mu) x d\mu, x^* \right\rangle \right| \\ &\leq 2K \text{Sup}_{|\mu| \geq N} \|H(\mu_0 + i\mu)\| \|x\|. \end{aligned}$$

Since $\lim_{|\mu| \rightarrow \infty} \|H(\mu_0 + i\mu)\| = 0$ there exists $N > 0$ such that

$$2K \text{Sup}_{|\mu| \geq N} \|H(\mu_0 + i\mu)\| < \epsilon$$

which yields the desired estimate $I_1(N) < \epsilon \|x\|$ for every $x \in D(A)$.

In order to estimate $I_2(N)$ we observe that $|e^{-i\alpha} - 1|^2 = 4\sin^2(\alpha/2)$, $\alpha \in \mathbb{R}$. Therefore, for the above fixed N we have

$$\begin{aligned} & \int_{|\mu| \leq N} |e^{-i\mu(s-t)} - 1| \|H''(\mu_0 + i\mu)x\| d\mu \\ & \leq 2 \int_{|\mu| \leq N} \left| \sin\left(\frac{(s-t)\mu}{2}\right) \right| \|H''(\mu_0 + i\mu)x\| d\mu \\ & \leq |s-t|N \int_{|\mu| \leq N} \|H''(\mu_0 + i\mu)x\| d\mu. \end{aligned}$$

Using these estimates for $I_1(N)$, $I_2(N)$ and the fact that $D(A)$ is dense in H yields $\|t^2 S_{\mu_0}(t) - s^2 S_{\mu_0}(s)\| < 2\epsilon$ for all $|s-t| < \delta$. This completes the proof. $\square$

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