REMARKS ON BLOW-UP BEHAVIOR FOR A NONLINEAR DIFFUSION EQUATION WITH NEUMANN BOUNDARY CONDITIONS

KENG DENG AND MINGXI XU

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ABSTRACT. We establish the blow-up rate for the solution of a nonlinear diffusion equation $(u^m)_t = u_{xx}$, $0 < x < 1$, $t > 0$, subject to Neumann boundary conditions $u_x(0, t) = 0$, $u_x(1, t) = u^\alpha(1, t)$.

1. INTRODUCTION

In this paper, we consider the initial-boundary value problem

$$(1.1) \quad \begin{aligned} (u^m)_t &= u_{xx}, & 0 < x < 1, \quad t > 0, \\ u_x(0, t) &= 0, \quad u_x(1, t) = u^\alpha(1, t), & t > 0, \\ u(x, 0) &= u_0(x) \geq \delta > 0, & 0 \leq x \leq 1. \end{aligned}$$

Here $0 < m$, $\alpha < \infty$, $u_0 \in C^{2+\nu}([0, 1])$ for some $0 < \nu < 1$, and $u'_0(0) = 0$, $u'_0(1) = u_0^\alpha(1)$.

Such a problem was previously studied in [5], wherein the following results were established.

Case 1. $0 < m < 1$. If $\alpha \leq m$, every solution of problem (1.1) exists globally while the solution blows up in a finite time T for $\alpha > m$. Moreover, if $m < \alpha \leq 1$, blow-up occurs on the whole interval $[0, 1]$, and there exist two positive constants C_1 and C_2 such that

$$(1.2) \quad C_1(T - t)^{1/(m-\alpha)} \leq u(x, t) \leq C_2(T - t)^{1/(m-\alpha)} \quad \text{for } (x, t) \in [0, 1] \times [0, T).$$

If $m < 1 < \alpha$, blow-up occurs only at $x = 1$.

Case 2. $m > 1$. If $2\alpha < m + 1$, problem (1.1) possesses a global solution, whereas for $2\alpha > m + 1$, every solution must blow up in a finite time T . Furthermore, if $\alpha > m$, blow-up occurs only at $x = 1$. And for $2\alpha > m + 1$, under an additional condition, the solution u satisfies

$$(1.3) \quad C_\epsilon(T - t)^{1/(m+1-2\alpha-\epsilon)} \leq u(1, t) \leq C(T - t)^{1/(m+1-2\alpha)} \quad \text{for } t \in [0, T)$$

with $\epsilon \in (0, 1)$ and $C_\epsilon, C > 0$. The condition is as follows.

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(H) For some $\lambda > 0$, $u_0(x) \in C^\infty([0, 1])$ is a positive solution of the problem

$$\begin{aligned} f''(x) &= \lambda m f^{m+\alpha-1}(x) \quad 0 < x < 1, \\ f'(0) &= 0, \quad f'(1) = f^\alpha(1). \end{aligned}$$

Later, it was shown in [8] that the critical case $2\alpha = m + 1$ belongs to the global existence part. When $m = 1$, the nonlinear diffusion equation reduces to the heat equation, and (1.1) has been extensively studied (see [4], [6] and the literature cited therein).

Several questions remain open: What is the blow-up rate for the case $m < 1 < \alpha$? Compared to the result for $m = 1$ in [4], can the lower bound in (1.3) be improved? And where does blow-up occur for the case $(m + 1)/2 < \alpha \leq m$?

Our objective in this paper is to answer all the above questions. For convenience, letting $a(u) = (1/m)u^{1-m}$ and $g(u) = u^\alpha$, in the sequel we shall mainly concentrate on the equivalent problem

$$\begin{aligned} (1.4) \quad u_t &= a(u)u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0, t) &= 0, \quad u_x(1, t) = g(u), & 0 < t < T, \\ u(x, 0) &= u_0(x), & 0 \leq x \leq 1. \end{aligned}$$

2. BLOW-UP RATE FOR THE CASE $m < 1 < \alpha$

In this section we establish bounds on the blow-up rate for $m < 1 < \alpha$. We first present the lower bound.

Theorem 2.1. *Let $m < 1 < \alpha$ and $u'_0(x) \geq 0$ on $[0, 1]$. Then there exists a positive constant C_3 such that*

$$(2.1) \quad u(1, t) \geq C_3(T - t)^{1/(m+1-2\alpha)} \quad \text{for } t \in [0, T).$$

Proof. Since $u'_0(x) \geq 0$ for $0 \leq x \leq 1$, the maximum principle yields that $u_x(x, t) \geq 0$ on $[0, 1] \times [0, T)$. Inspired by [1], [3], we define a function $\Phi(x, t) = u_x(x, t) - \varphi(x)g(u(x, t))$ in $(0, 1) \times (0, T)$, where $\varphi(x)$ is a nonnegative function to be determined. Through a routine calculation, we obtain

$$\begin{aligned} (2.2) \quad \Phi_t &= a(u)\Phi_{xx} + a'(u)u_x\Phi_x \\ &\quad + \varphi(x)(a'(u)g'(u) + a(u)g''(u))u_x^2 \\ &\quad + \varphi'(x)(a'(u)g(u) + 2a(u)g'(u))u_x + \varphi''(x)a(u)g(u). \end{aligned}$$

Since $a'(u)g'(u) + a(u)g''(u) = (\alpha/m)(\alpha - m)u^{\alpha-m-1} > 0$ and $a'(u)g(u) + 2a(u)g'(u) = ((2\alpha - m + 1)/m)u^{\alpha-m} > 0$, if $\varphi(x)$ satisfies $\varphi' \geq 0$ and $\varphi'' \geq 0$ on $(0, 1)$, then Φ cannot attain its negative minimum in $(0, 1) \times (0, T)$.

Because $u'_0(1) = u_0^\alpha(1)$, $u'_0(x) > 0$ near $x = 1$, we can construct $\varphi(x)$ as follows: $\varphi(x) \equiv 0$ for $0 \leq x \leq x_0$ with some $x_0 < 1$ and $\varphi(x) = (x - x_0)^l / (1 - x_0)^l$ on $(x_0, 1]$, where $l (\geq 3)$ is chosen so large that $\varphi(x) \leq u'_0(x)u_0^{-\alpha}(x)$ for $x_0 < x \leq 1$. Then $\Phi(0, t) = \Phi(1, t) = 0$, and $\Phi(x, 0) \geq 0$. By the maximum principle, we have $\Phi(x, t) \geq 0$, that is,

$$(2.3) \quad u_x(x, t) \geq \varphi(x)g(u(x, t)) \quad \text{for } (x, t) \in [0, 1] \times [0, T).$$

Moreover, $\Phi_x(1, t) \leq 0$, which means

$$\begin{aligned} (2.4) \quad u_t(1, t) &= a(u(1, t))u_{xx}(1, t) \\ &\leq a(u(1, t))g(u(1, t))(\varphi'(1) + g'(u(1, t))) \leq c_0 u^{2\alpha-m}(t). \end{aligned}$$

Integrating (2.4) from t to T yields the asserted result.

We then give the upper bound, the derivation of which is in the spirit of [2].

Theorem 2.2. *Let $m < 1 < \alpha$ and $u_0'(x), u_0''(x) \geq 0$ on $[0, 1]$. Then there exists a positive constant C_4 such that*

$$(2.5) \quad u(1, t) \leq C_4(T - t)^{1/(m+1-2\alpha)} \quad \text{for } t \in [0, T].$$

Proof. Since $u_0''(x) \geq 0$ for $0 \leq x \leq 1$, by the maximum principle one can see that $u_{xx}(x, t)$ and $u_t(x, t)$ are positive on $[0, 1] \times [\tau, T)$ for any $\tau \in (0, T)$. Letting τ be close to T , consider $\Psi(x, t) = u_t(x, t) - \epsilon d(u(x, t))u_x^{2-\sigma}(x, t)$ in $(1 - T + \tau, 1) \times (\tau, T)$, where ϵ is a positive constant, $(m + 1)/\alpha < \sigma < (m + \alpha)/\alpha$, and $d(u) = a(u)g'(u)g^{\sigma-1}(u)$. By a tedious computation we find that

$$\Psi_t = a(u)\Psi_{xx} + C(x, t)\Psi + \epsilon R(x, t)u_x^{4-\sigma},$$

where

$$(2.6) \quad \begin{aligned} C(x, t) = & \left[\frac{a'(u)}{a(u)} + \epsilon(2 - \sigma)(1 - \sigma) \frac{d(u)}{a(u)} u_x^{-\sigma} \right] u_t \\ & + \epsilon \left[(4 - 2\sigma)d'(u) - (1 - \sigma) \frac{a'(u)d(u)}{a(u)} \right] u_x^{2-\sigma} \\ & + \epsilon^2(2 - \sigma)(1 - \sigma) \frac{d^2(u)}{a(u)} u_x^{2-2\sigma} \end{aligned}$$

and

$$(2.7) \quad \begin{aligned} R(x, t) = & a(u)d''(u) + \epsilon[(\sigma - 1)a'(u)d(u) + 2(2 - \sigma)a(u)d'(u)] \frac{d(u)}{a(u)} u_x^{-\sigma} \\ & + \epsilon^2(2 - \sigma)(1 - \sigma) \frac{d^3(u)}{a(u)} u_x^{-2\sigma}. \end{aligned}$$

Since $a(u) = (1/m)u^{1-m}$ and $d(u) = (\alpha/m)u^{\alpha\sigma-m}$, we have

$$(2.8) \quad \begin{aligned} R(x, t) = & \frac{\alpha}{m^2} \left\{ (\alpha\sigma - m)(\alpha\sigma - m - 1) + \epsilon\alpha[(\sigma - 1)(1 - m) \right. \\ & \left. + 2(2 - \sigma)(\alpha\sigma - m)] \frac{g^\sigma(u)}{u_x^\sigma} + \epsilon^2\alpha^2(2 - \sigma)(1 - \sigma) \frac{g^{2\sigma}(u)}{u_x^{2\sigma}} \right\} u^{\alpha\sigma-2m-1}. \end{aligned}$$

Recalling (2.3) we observe that if $1 - T + \tau > x_0$,

$$(2.9) \quad u_x(x, t) \geq kg(u(x, t)) \quad \text{for } (x, t) \in [1 - T + \tau, 1] \times [\tau, T)$$

with k a positive constant. Consequently, $R(x, t) \geq 0$ for small ϵ , which implies

$$(2.10) \quad \Psi_t \geq a(u)\Psi_{xx} + C(x, t)\Psi \quad \text{in } (1 - T + \tau, 1) \times (\tau, T).$$

On the parabolic boundary, since $x = 1$ is the only blow-up point, if ϵ is small enough, then both $\Psi(1 - T + \tau, t)$ and $\Psi(x, \tau)$ are nonnegative. At $x = 1$,

$$(2.11) \quad \Psi_x(1, t) = b(1, t)\Psi(1, t) + \epsilon g^{2-\sigma}(u(1, t))Q(u(1, t)),$$

where

$$b(x, t) = g'(u) - \epsilon(2 - \sigma) \frac{d(u)}{a(u)} g^{1-\sigma}(u)$$

and

$$\begin{aligned}
 Q(u) &= d(u)g'(u) - d'(u)g(u) - \epsilon(2 - \sigma)\frac{d^2(u)}{a(u)}g^{1-\sigma}(u) \\
 (2.12) \quad &= \frac{\alpha}{m}(\alpha + m - \alpha\sigma - \epsilon\alpha(2 - \sigma))u^{\alpha\sigma + \alpha - m - 1}.
 \end{aligned}$$

Since $\sigma < (m + \alpha)/\alpha$, if ϵ is sufficiently small, then $Q(u(1, t)) > 0$, which means that $\Psi_x(1, t) > b(1, t)\Psi(1, t)$ for $\tau \leq t < T$. Thus $\Psi(x, t) \geq 0$ on $[1 - T + \tau, 1] \times [\tau, T)$. In particular, $\Psi(1, t) \geq 0$, that is ,

$$(2.13) \quad u_t(1, t) \geq \epsilon a(u(1, t))g(u(1, t))g'(u(1, t)) = \frac{\epsilon\alpha}{m}u^{2\alpha - m}(1, t).$$

Integration of (2.13) over (t, T) then leads to (2.5).

3. BLOW-UP RATE AND SET FOR THE CASE $1 < m < 2\alpha - 1$

We begin this section by presenting the lower bound on the blow-up rate for $2\alpha > m + 1$. Here we use a modification of an argument from [5].

Theorem 3.1. *Let $1 < m < 2\alpha - 1$ and $u'_0(x), u''_0(x) \geq 0$ on $[0, 1]$. Then there exists a positive constant C_5 such that*

$$(3.1) \quad u(1, t) \geq C_5(T - t)^{1/(m+1-2\alpha)} \quad \text{for } t \in [0, T).$$

Proof. It suffices to prove (3.1) for $t \in [\eta, T)$ with some η such that $u(1, \eta) \geq 1$. Set

$$(3.2) \quad y(t) = u^\gamma(1, t) \int_{1-\xi(t)}^1 u^m(x, t) dx$$

with

$$(3.3) \quad \xi(t) = (1 - \epsilon)u^{1-\alpha}(1, t),$$

where $0 < \epsilon < 1$, $(1 - \epsilon)/\epsilon^m < 2/(\alpha - 1)$, and $\alpha - m - 1 < \gamma \leq (\alpha - 1)[1 - m(1 - \epsilon)/2\epsilon^m]$. A straightforward computation gives

$$\begin{aligned}
 y'(t) &= \gamma u^{\gamma-1}(1, t)u_t(1, t) \int_{1-\xi(t)}^1 u^m(x, t) dx \\
 &\quad + u^\gamma(1, t)(u_x(1, t) - u_x(1 - \xi(t), t)) \\
 (3.4) \quad &\quad + (1 - \epsilon)(1 - \alpha)u^{\gamma-\alpha}(1, t)u^m(1 - \xi(t), t)u_t(1, t) \\
 &\leq -u^{\gamma-1}(1, t)u_t(1, t)I(t) + u^{\gamma+\alpha}(1, t),
 \end{aligned}$$

where $I(t) = (\alpha - 1)u^m(1 - \xi(t), t)\xi(t) - \gamma \int_{1-\xi(t)}^1 u^m(x, t) dx$. Since $u_x \geq 0$ and $u_{xx} \geq 0$ in $[0, 1] \times [\eta, T)$, we find

$$(3.5) \quad \epsilon u(1, t) \leq u(1 - \xi(t), t) \leq u(x, t) \leq u(1, t)$$

for any $x \in [1 - \xi(t), 1]$ and $t \in [\eta, T)$. Moreover, by (3.2), (3.3), and (3.5), we have

$$(3.6) \quad (1 - \epsilon)\epsilon^m u^{\gamma+m+1-\alpha}(1, t) \leq y(t) \leq (1 - \epsilon)u^{\gamma+m+1-\alpha}(1, t),$$

or equivalently,

$$(3.7) \quad c_1 u(1, t) \leq y^{1/(\gamma+m+1-\alpha)}(t) \leq c_2 u(1, t) \quad \text{for } t \in [\eta, T).$$

We now show that $I(t) \geq 0$ on $[\eta, T)$.

$$\begin{aligned}
 I(t) &= (1-\alpha) \int_{1-\xi(t)}^1 (u^m(x, t) - u^m(1-\xi(t), t)) dx \\
 &\quad + (\alpha - \gamma - 1) \int_{1-\xi(t)}^1 u^m(x, t) dx \\
 (3.8) \quad &= (1-\alpha) \int_{1-\xi(t)}^1 m u^{m-1}(\zeta(t), t) u_x(\zeta(t), t) (x + \xi(t) - 1) dx \\
 &\quad + (\alpha - \gamma - 1) \int_{1-\xi(t)}^1 u^m(x, t) dx,
 \end{aligned}$$

where $1 - \xi(t) < \zeta(t) < 1$. Because $\alpha, m > 1$, $\alpha - \gamma - 1 > 0$, $u_x, u_{xx} \geq 0$, and $\int_{1-\xi(t)}^1 (x + \xi(t) - 1) dx = (1/2)(1 - \epsilon)^2 u^{2-2\alpha}(1, t)$, we can see that

$$(3.9) \quad I(t) \geq (1 - \epsilon) \left[\frac{1}{2} m(1 - \alpha)(1 - \epsilon) + (\alpha - \gamma - 1) \epsilon^m \right] u^{m+1-\alpha}(1, t) \geq 0.$$

From (3.4), (3.7), and (3.9), it then follows that

$$(3.10) \quad y'(t) \leq u^{\gamma+\alpha}(1, t) \leq c_1^{-\gamma-\alpha} y^{(\gamma+\alpha)/(\gamma+m+1-\alpha)}(t),$$

since $\gamma + \alpha > 2\alpha - m - 1 > 0$. Integrating the above inequality from t to T , we obtain

$$y^{(m+1-2\alpha)/(\gamma+m+1-\alpha)}(t) \leq \frac{2\alpha - m - 1}{\gamma + m + 1 - \alpha} c_1^{-\gamma-\alpha} (T - t),$$

that is,

$$y^{1/(\gamma+m+1-\alpha)}(t) \geq c_3 (T - t)^{1/(m+1-2\alpha)},$$

which in conjunction with (3.7) leads to (3.1).

Based on the general idea of [7], we then locate the blow-up point for the case $(m+1)/2 < \alpha \leq m$.

Theorem 3.2. *Suppose that condition (H) holds. Then for $(m+1)/2 < \alpha \leq m$, blow-up occurs only at $x = 1$.*

Proof. Introduce a function $w(x) \in C^2([0, 1])$ defined by

$$(3.11) \quad w(x) = \begin{cases} h(x) & 0 \leq x \leq \frac{1}{2}, \\ (1-x)^2 & \frac{1}{2} < x \leq 1 \end{cases}$$

with $w'(0) = 0$ and $w'(x) \leq 0$ for $0 < x \leq 1$. Let $v(x, t) = C_0[w(x) + B(T - t)]^{-\rho}$, where C_0 and B are positive constants to be determined, and $\rho = 1/(2\alpha - m - 1)$. Then $v(x, t)$ satisfies

$$(3.12) \quad m u^{m-1} v_t - v_{xx} = C_0 \rho [w(x) + B(T - t)]^{-\rho-1} J(x, t) \quad \text{in } (0, 1) \times (0, T).$$

Here $J(x, t) = B m u^{m-1} - (\rho + 1)(w'(x))^2 / [w(x) + B(T - t)] + w''(x)$. Since $u_0(x) \geq \delta > 0$ and $u_t \geq 0$, for $(x, t) \in (1/2, 1) \times (0, T)$,

$$\begin{aligned}
 J(x, t) &= B m u^{m-1} - \frac{4(\rho + 1)(1-x)^2}{(1-x)^2 + B(T-t)} + 2 \\
 (3.13) \quad &\geq B m \delta^{m-1} - 4\rho - 2 \geq 0
 \end{aligned}$$

if $B \geq (4\rho + 2)/m\delta^{m-1}$; for $(x, t) \in (0, 1/2] \times (0, T)$,

$$(3.14) \quad J(x, t) \geq Bm\delta^{m-1} - \frac{(\rho + 1)h_1^2}{h(\frac{1}{2})} - h_2 \geq 0$$

if $B \geq [(\rho + 1)h_1^2/h(1/2) + h_2]/m\delta^{m-1}$, where $h_1 = \max_{0 \leq x \leq \frac{1}{2}} |h'(x)|$ and $h_2 = \max_{0 \leq x \leq \frac{1}{2}} |h''(x)|$.

On the parabolic boundary, $v_x(0, t) = u_x(0, t) = 0$. By (1.3), if $C_0 \geq B^\rho C$, then $v(1, t) = C_0 B^{-\rho} (T - t)^{-\rho} \geq u(1, t)$; and when $t = 0$, C_0 can be chosen so large that

$$v(x, 0) \geq C_0[w(0) + BT]^{-\rho} \geq \max_{0 \leq x \leq 1} u_0(x) \geq u_0(x).$$

Then the maximum principle implies

$$u(x, t) \leq v(x, t) \quad \text{for } (x, t) \in [0, 1] \times [0, T),$$

which shows that for any $x \in [0, 1)$,

$$u(x, t) \leq C_0 w^{-\rho}(x) < \infty.$$

Thus the proof is completed.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHWESTERN LOUISIANA, LAFAYETTE, LOUISIANA 70504

E-mail address: kxd5858@usl.edu

E-mail address: mxx3473@usl.edu