TOEPLITZ C^* -ALGEBRAS ON ORDERED GROUPS AND THEIR IDEALS OF FINITE ELEMENTS

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ABSTRACT. Let G be a discrete abelian group and (G, G_+) an ordered group. Denote by (G, G_F) the minimal quasily ordered group containing (G, G_+) . In this paper, we show that the ideal of finite elements is exactly the kernel of the natural morphism between these two Toeplitz C^* -algebras. When G is countable, we show that if the direct sum of K -groups $K_0(\mathcal{T}^{G_+}) \oplus K_1(\mathcal{T}^{G_+}) \cong \mathbb{Z}$, then $K_0(\mathcal{T}^{G_+}) \cong \mathbb{Z}$.

1. PRELIMINARIES AND THE MAIN RESULT

Let G be a discrete abelian group and G_+ a subset of G , (G, G_+) is said to be a quasily ordered group, if $0 \in G_+$, $G_+ + G_+ \subseteq G_+$, and $G = G_+ \cup (-G_+)$. If in addition, $G_+^0 = G_+ \cap (-G_+) = \{0\}$, then (G, G_+) is referred to as an ordered group.

In this section, we always fix an ordered group (G, G_+) . Let $\widehat{G}$ be the dual group of G , $\widehat{G}$ is compact and it is connected since G is torsion-free. When given the normal Haar measure, it is easy to show that $\{\varepsilon_x \mid x \in G\}$ is an orthonormal basis for $L^2(\widehat{G})$, where ε_x is defined by $\varepsilon_x(\gamma) = \gamma(x)$ for $\gamma \in \widehat{G}$.

For any $E \subseteq G$, let $H^2(E)$ be the closed subspace of $L^2(\widehat{G})$ generated by $\{\varepsilon_x \mid x \in E\}$; its projection is denoted by p^E . For any $\varphi \in C(\widehat{G})$, define T_φ^E on $H^2(E)$ by $T_\varphi^E(f) = p^E(\varphi f)$ for $f \in H^2(E)$. We say that T_φ^E is a Toeplitz operator (relative to E) with symbol φ . The C^* -algebra generated by $\{T_\varphi^E \mid \varphi \in C(\widehat{G})\}$ is denoted by $\mathcal{T}_r^E$, and is called the Toeplitz C^* -algebra with respect to E . By the Stone-Weierstrass theorem, $\mathcal{T}_r^E$ is also generated by $\{T_{\varepsilon_x}^E \mid x \in G\}$.

Now let (G, G_+) be an ordered group, denote by $F(G) = F(G_+) \cup (-F(G_+))$, where

$$F(G_+) = \{x \in G_+ \mid \forall y \in G_+ \setminus \{0\}, \exists n \in \mathbb{N}, \text{ such that } ny - x \in G_+\}.$$

It is easy to show that $F(G)$ is a subgroup of G and its elements are usually called the finite elements. Denote by $K(F(G))$, the closed two-sided ideal of $\mathcal{T}_r^{G_+}$

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generated by

$$\{1 - T_{\varepsilon_x}^{G_+} T_{\varepsilon_{-x}}^{G_+} \mid x \in F(G_+)\}.$$

Let $G_F = G_+ \cup F(G)$, then obviously $G_F^0 = F(G)$, and it is easy to show that

$$G_F = \bigcap_{g \in G_+ \setminus \{0\}} (G_+ - Z_+ g) = G_+ - F(G_+).$$

So when $F(G) \neq \{0\}$, (G, G_F) is the minimal quasily ordered group containing (G, G_+) .

A representation of an ordered group (G, G_+) by isometries on a Hilbert space H , is a map $V : G_+ \rightarrow \mathbb{B}(H)$ such that

$$V(0) = 1, V(x)^* V(x) = 1 \text{ and } V(x)V(y) = V(x+y) \text{ for all } x, y \in G_+.$$

The Toeplitz C^* -algebra $\mathcal{T}_r^{G_+}$ has a universal property for isometric representations, that is, if (V, H) is any isometric representation of G_+ , then there is a unique C^* -morphism $\pi_V : \mathcal{T}_r^{G_+} \rightarrow \mathbb{B}(H)$, such that $\pi_V(T_{\varepsilon_x}^{G_+}) = V(x)$ for all $x \in G_+$. Furthermore, if every $V(x), x \neq 0$ is non-unitary, then π_V is isometric (see [1], Theorem 1.3 and Theorem 2.9 or [5], Section 5.2). So for any quasily ordered group (G, E) with $G_+ \subseteq E$, there is a C^* -algebra morphism $\gamma^{E, G_+} : \mathcal{T}_r^{G_+} \rightarrow \mathcal{T}_r^E$ such that

$$\gamma^{E, G_+}(T_{\varepsilon_g}^{G_+}) = T_{\varepsilon_g}^E \text{ for all } g \in G_+.$$

Obviously, $K(F(G)) \subseteq \text{Ker } \gamma^{G_F, G_+}$. The main result of this paper is, we prove that $K(F(G))$ is equal to $\text{Ker } \gamma^{G_F, G_+}$.

Remark. By an isometric representation (V, H) of a quasily ordered group (G, G_F) , we mean H is a Hilbert space and $V : G_F \rightarrow \mathbb{B}(H)$ is a map satisfying

$$\begin{aligned} V(0) = 1, V^*(x)V(x) = 1, V(x+y) = V(x)V(y) \text{ for all } x, y \in G_F; \\ V(x)V(x)^* = 1 \text{ for all } x \in F(G). \end{aligned}$$

The following lemma is crucial to our proof of Theorem 1.2, we defer its proof to Section 3.

Lemma 1.1. *Any isometric representation (V, H) of G_F on a Hilbert space H can be lifted to be a C^* -algebra morphism $\pi_V : \mathcal{T}_r^{G_F} \rightarrow \mathbb{B}(H)$, such that $\pi_V(T_{\varepsilon_x}^{G_F}) = V(x)$ for all $x \in G_F$.*

Theorem 1.2. *Let (G, G_+) be an ordered group and $F(G)$ the subgroup of finite elements, then $K(F(G)) = \text{Ker } \gamma^{G_F, G_+}$.*

Proof. It needs only to prove “ $\text{Ker } \gamma^{G_F, G_+} \subseteq K(F(G))$ ”. Let ρ be the natural morphism induced by γ^{G_F, G_+} from $\mathcal{T}_r^{G_+}/K(F(G))$ to $\mathcal{T}_r^{G_F}$, then by Lemma 1.1 ρ has an inverse π_V lifted by the isometric representation of G_F on $\mathcal{T}_r^{G_+}/K(F(G))$ defined by

$$V(x) = [T_{\varepsilon_x}^{G_+}] \in \mathcal{T}_r^{G_+}/K(F(G)) \text{ for } x \in G_F,$$

therefore, $\text{Ker } \gamma^{G_F, G_+} \subseteq K(F(G))$. $\square$

Let (G, G_+) be as above and let V be a representation of G_+ by isometries. If for every $x \in G_+$ with $x \neq 0$, $V(x)$ is non-unitary, then $\mathcal{T}_r^{G_+} \cong C^*(\{V(x) \mid x \in G_+\})$. On the other hand, if there does exist some $y \in G_+, y \neq 0$ such that $V(y)$ is a

unitary, then it is easy to show that V is also an isometric representation of G_F (when $x \notin G_+$, we define $V(x) = V(-x)^*$). Indeed, for any $x \in F(G_+)$, by definition there is an $n \in N$ such that $ny - x \in G_+$. So

$$V(ny - x)V(x)V(x)^*V(ny - x)^* = V(ny)V(ny)^* = V(y)^n(V(y)^n)^* = 1.$$

Multiply with $V(ny - x)^*$ on the left and $V(ny - x)$ on the right, we obtain $V(x)V(x)^* = 1$. Therefore, we have the following

Corollary 1.3. *Let (G, G_+) be an ordered group with $F(G) \neq \{0\}$. Then every non-faithful representation π of $\mathcal{T}_r^{G_+} \rightarrow \mathbb{B}(H)$ can be factored through $\mathcal{T}_r^{G_F}$, in the sense that there exists a representation $\pi_F : \mathcal{T}_r^{G_F} \rightarrow \mathbb{B}(H)$ such that $\pi = \pi_F \circ \gamma^{G_F, G_+}$.*

Remark. (1) Let (G, G_+) be an ordered group and $F(G)$ the subgroup of finite elements. By Corollary 1.3 and Theorem 1.2, it is easy to show that $K(F(G))$ is contained in any closed two-sided ideal of $\mathcal{T}^{G_+}$. In particular, $K(F(G))$ is simple and $\mathcal{T}_r^{G_+}$ is prime. Such a property was also established by G. Murphy in a far different way (see [1], Theorem 2.10 and Theorem 2.11). Our proof here in some sense is sensibly simpler than the original one.

(2) Suppose $F(G) \neq \{0\}$. If G admits a least positive element, equivalently, $\mathcal{T}_r^{G_+}$ contains a Fredholm operator of non-zero index, or $K(F(G))$ is the ideal of compact operators on $H^2(G_+)$ ([2], Lemma 2.2), then by Theorem 1.2 we know that

$$\forall T \in \mathcal{T}_r^{G_+}, T \text{ is Fredholm if and only if } \gamma^{G_F, G_+}(T) \text{ is invertible in } \mathcal{T}_r^{G_F}.$$

When $(G, G_+) = (\mathbb{Z}, \mathbb{Z}_+)$, it reduces obviously to the classical case. A precise character of a single Toeplitz operator with continuous symbol to be Fredholm can be found in ([2], Theorem 2.5 and Theorem 4.2), and a generalized version of [2] has been given in [8].

2. A CHARACTER OF K -GROUPS OF TOEPLITZ C^* -ALGEBRAS ON ORDERED GROUPS

Let A be a C^* -algebra, denote by $K_0(A)$ and $K_1(A)$ the K_0 -group and K_1 -group of A respectively. If $K_0(A) \oplus K_1(A) \cong \mathbb{Z}$, then obviously either $K_0(A) \cong \mathbb{Z}, K_1(A) = 0$ or $K_1(A) \cong \mathbb{Z}, K_0(A) = 0$. Let $\mathcal{T}$ be the classical Toeplitz C^* -algebra, it is well-known that $K_0(\mathcal{T}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$. On the other hand, there are many C^* -algebras, such as $\mathbb{B}/\mathbb{K}, C_0(\mathbb{R}^{2n+1})$, their K_1 -groups are all isomorphic to $\mathbb{Z}$ and their K_0 -groups are all equal to zero. Quite surprisingly, in this section, we show that when G is a countably infinite discrete abelian group and (G, G_+) is an ordered group, if the direct sum of the K -groups $K_0(\mathcal{T}_r^{G_+}) \oplus K_1(\mathcal{T}_r^{G_+})$ is isomorphic to $\mathbb{Z}$, then $K_0(\mathcal{T}_r^{G_+})$ must be isomorphic to $\mathbb{Z}$ (of course in this case $K_1(\mathcal{T}_r^{G_+})$ must be equal to zero).

In this section, we always assume G is a countably infinite discrete abelian group. Let (G, G_+) be an ordered group, a subgroup I of G is said to be an ideal, if $0 \leq x \leq y \in I \cap G_+$ implies $x \in I$ for all $x \in G$, where “ $\leq$ ” is the usual order on G induced by G_+ . For any $x \in G$, set $V_x = T_{\varepsilon_x}^{G_+} \in \mathcal{T}_r^{G_+}$. Let I be an ideal of G , we define $\mathcal{T}(G, I)$ to be the closed ideal in $\mathcal{T}_r^{G_+}$ generated by $\{1 - V_x V_x^* \mid x \in I \cap G_+\}$. If $(G_1, (G_1)_+), \dots, (G_n, (G_n)_+)$ are ordered groups, we denote by $(G_1 \times \dots \times G_n, G_1 * \dots * G_n)$ the *lexicographic-ordered* group.

Lemma 2.1 ([3]). *Let $(G_1, (G_1)_+)$ and $(G_2, (G_2)_+)$ be two ordered groups. Then there is a C^* -morphism $\alpha : \mathcal{T}_r^{G_1 * G_2} \rightarrow C(\widehat{G_2}) \otimes \mathcal{T}_r^{G_1}$ such that for all $(x, y) \in G_1 * G_2$, we have $\alpha(V_{(x,y)}) = \varepsilon_y \otimes V_x$. Moreover $\text{Ker } \alpha = \mathcal{T}(G_1 \times G_2, G_2)$.*

Theorem 2.2. *Let (G, G_+) be an ordered group. If $K_0(\mathcal{T}_r^{G+}) \oplus K_1(\mathcal{T}_r^{G+}) \cong \mathbb{Z}$, then $K_0(\mathcal{T}_r^{G+}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}_r^{G+}) = 0$.*

Proof. We identify $\{0\} \times \mathbb{Z}$ with $\mathbb{Z}$. By the above lemma, we have the following short exact sequence of C^* -algebras:

$$0 \longrightarrow \mathcal{T}(G \times \mathbb{Z}, \mathbb{Z}) \xrightarrow{i} \mathcal{T}_r^{G * \mathbb{Z}} \xrightarrow{\alpha} C(T) \otimes \mathcal{T}_r^{G+} \longrightarrow 0$$

where i is the inclusion map and $\alpha(V_{(x,n)}) = \varepsilon_n \otimes V_x$ for any $(x, n) \in G * \mathbb{Z}$. Since the subgroup of finite elements in $G \times \mathbb{Z}$ is just $\mathbb{Z}$, by ([2], Lemma 2.2) we know that $\mathcal{T}(G \times \mathbb{Z}, \mathbb{Z}) = \mathbb{K}(H^2(G * \mathbb{Z}))$, where $\mathbb{K}(H^2(G * \mathbb{Z}))$ is the ideal of compact operators on $H^2(G * \mathbb{Z})$. So we have the following periodic six-term exact sequence of K -theory:

$$\begin{array}{ccccc} \mathbb{Z} \cong K_0(\mathbb{K}) & \xrightarrow{i_*} & K_0(\mathcal{T}_r^{G * \mathbb{Z}}) & \xrightarrow{\alpha_*} & K_0(C(T) \otimes \mathcal{T}_r^{G+}) \\ \uparrow \text{index map} & & & & \downarrow \delta \\ K_1(C(T) \otimes \mathcal{T}_r^{G+}) & \xleftarrow{\alpha_*} & K_1(\mathcal{T}_r^{G * \mathbb{Z}}) & \xleftarrow{i_*} & K_1(\mathbb{K}) = 0 \end{array}$$

Since $C(T) \otimes \mathcal{T}_r^{G+} \cong C(T \rightarrow \mathcal{T}_r^{G+})$, by ([7], Exercise 8.B) (or simply use the Künneth theorem) we know that $K_0(C(T) \otimes \mathcal{T}_r^{G+}) \cong K_1(C(T) \otimes \mathcal{T}_r^{G+}) \cong K_0(\mathcal{T}_r^{G+}) \oplus K_1(\mathcal{T}_r^{G+}) \cong \mathbb{Z}$. If we also denote the index map by δ , then since $V_{(0,1)}$ is an isometry, by the definition of index map ([7], Definition 8.1.1) and ([7], Exercise 8.C) we know that $\delta[z \otimes 1_{\mathcal{T}_r^{G+}}] = -[1 - V_{(0,1)} V_{(0,1)}^*] \in K_0(\mathbb{K})$, where $z \in C(T)$ is the inclusion of T into $\mathbb{R}$. Note that any homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is injective iff it is non-zero, and it is surjective iff 1 or -1 is in its range. Obviously, the rank of $1 - V_{(0,1)} V_{(0,1)}^*$ is 1, so the index map in the exact sequence is an isomorphism; therefore,

$$K_0(\mathcal{T}_r^{G * \mathbb{Z}}) \cong \mathbb{Z}, \quad K_1(\mathcal{T}_r^{G * \mathbb{Z}}) = 0.$$

The map

$$G * \mathbb{Z} \rightarrow \mathcal{T}_r^{G+}, \quad (x, n) \rightarrow V_x,$$

is an isometric homomorphism, and therefore induces a C^* -morphism $\rho : \mathcal{T}_r^{G * \mathbb{Z}} \rightarrow \mathcal{T}_r^{G+}$ such that $\rho(V_{(x,n)}) = V_x$ for all $(x, n) \in G * \mathbb{Z}$. Similarly, there is a C^* -morphism $\theta : \mathcal{T}_r^{G+} \rightarrow \mathcal{T}_r^{G * \mathbb{Z}}$ such that $\theta(V_x) = V_{(x,0)}$ for all $x \in G_+$. Since $\mathcal{T}_r^{G+}$ is generated by $\{V_x \mid x \in G_+\}$, we know that $\rho \circ \theta = id_{\mathcal{T}_r^{G+}}$, so $\rho_* \circ \theta_* = id_{K_1(\mathcal{T}_r^{G+})}$, it follows that the map $\rho_* : K_1(\mathcal{T}_r^{G * \mathbb{Z}}) \rightarrow K_1(\mathcal{T}_r^{G+})$ is onto, so $K_1(\mathcal{T}_r^{G+}) = 0$. $\square$

Remark. It is easy to show by induction that for any $n \in \mathbb{N}$, $K_0(\mathcal{T}^{\mathbb{Z} * \dots * \mathbb{Z}}) \cong \mathbb{Z}$ and $K_1(\mathcal{T}^{\mathbb{Z} * \dots * \mathbb{Z}}) = 0$, and this was established in [4]. Next we give another Toeplitz C^* -algebra, whose K_0 -group is also isomorphic to $\mathbb{Z}$ and whose K_1 -group is trivial.

Example. Let

$$G = \mathbb{Z}^2, G_+ = \{(m, n) \in \mathbb{Z}^2 \mid m + n > 0, \text{ or } m + n = 0 \text{ and } m \geq 0\}.$$

Clearly, (G, G_+) is an ordered group, $(1, -1)$ is the least positive element of G and

$$F(G) = \{ (n, -n) \mid n \in \mathbb{Z} \}.$$

So

$$G_F = \{ (m, n) \in \mathbb{Z}^2 \mid m + n \geq 0 \}.$$

By Theorem 1.2, we have the following short exact sequence of C^* -algebras:

$$0 \longrightarrow \mathbb{K}(H^2(G_+)) \xrightarrow{i} \mathcal{T}_r^{G_+} \xrightarrow{\gamma^{G_F, G_+}} \mathcal{T}_r^{G_F} \longrightarrow 0.$$

Therefore, we have the following six-term exact sequence of K -theory:

$$\begin{array}{ccccc} \mathbb{Z} \cong K_0(\mathbb{K}) & \xrightarrow{i_*} & K_0(\mathcal{T}_r^{G_+}) & \xrightarrow{(\gamma^{G_F, G_+})_*} & K_0(\mathcal{T}_r^{G_F}) \\ \uparrow \text{index map} & & & & \downarrow \delta \\ K_1(\mathcal{T}_r^{G_F}) & \xleftarrow{(\gamma^{G_F, G_+})_*} & K_1(\mathcal{T}_r^{G_+}) & \xleftarrow{i_*} & K_1(\mathbb{K}) = 0 \end{array}$$

By ([6], Section 3 (set $\alpha = -1$)), we know that $\mathcal{T}_r^{G_F} \cong C(T) \otimes \mathcal{T}$, where $\mathcal{T}$ is the classical Toeplitz C^* -algebra, so $K_0(\mathcal{T}_r^{G_F}) \cong \mathbb{Z} \cong K_1(\mathcal{T}_r^{G_F})$. As before we know that the index map is an isomorphism, therefore we know that

$$K_0(\mathcal{T}_r^{G_+}) \cong \mathbb{Z}, \quad K_1(\mathcal{T}_r^{G_+}) = 0.$$

3. THE PROOF OF LEMMA 1.1

The ideals of this section are mostly contained in [5]. Roughly speak, the quasi-lattice ordered groups discussed in [5] are “ordered groups”. What we consider here are the quasily ordered groups; in other words, the unit space may be not trivial.

We deal with Toeplitz C^* -algebras in another context.

Let G be a discrete (not necessary to be abelian) group and P a subsemigroup of G . Then (G, P) is said to be a quasily ordered group, if

$$e \in P, P \cdot P \subseteq P, \text{ and } G = P \cup P^{-1},$$

where e is the unit of G and $P^{-1} = \{ x^{-1} \mid x \in P \}$.

Now let (G, P) be a quasily ordered group, denote by $G_+^0 = P \cap P^{-1}$, it is a subgroup of G . Since $(P \setminus G_+^0) \cdot P = (P \setminus G_+^0) = P \cdot (P \setminus G_+^0)$, if we let $G_+ = (P \setminus G_+^0) \cup \{e\}$, then $G = G_+ \cup G_+^0 \cup (G_+)^{-1}$ and (G, G_+) is a partially ordered group in the sense that

$$e \in G_+, G_+ \cdot G_+ \subseteq G_+, G_+ \cap G_+^{-1} = \{e\}, \text{ and } G = G_+ \cdot (G_+)^{-1}.$$

For any $x, y \in G$, we say

$$x \leq y \text{ (resp. } x \ll y, x \sim y) \text{ if } x^{-1}y \in G_+ \text{ (resp. } x^{-1}y \in P, x^{-1}y \in G_+^0).$$

Let $\{\delta_g \mid g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$\delta_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases} \quad \text{for } g, h \in G.$$

For any quasily ordered group (G, P) , another Toeplitz C^* -algebra $W(G, P)$ can be defined as the C^* -subalgebra of $\mathbb{B}(\ell^2(P))$ generated by $\{J^*L_tJ \mid t \in G\}$ with $L : G \rightarrow \mathbb{B}(\ell^2(G))$ the left regular representation and $J : \ell^2(P) \rightarrow \ell^2(G)$ the inclusion operator. For any $t \in P$, the operator J^*L_tJ , which will now be denoted by $w(t)$ and let $w(t)w(t)^* = m(t)$ as in [5] for any $t \in P$.

The formulas listed in Proposition 3.1 below are clear:

Proposition 3.1. (1) $w(s)w(t) = w(st), w(t)^*w(t) = 1, w(s) = w(t) \Leftrightarrow s = t, \forall s, t \in P$.

(2) $w(s_1)w(t_1)^* = w(s_2)w(t_2)^* \Leftrightarrow t_1 \sim t_2$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$.

(3) $w(s)^*w(t) = \begin{cases} w(s^{-1}t), & \text{if } s \ll t, \\ w(t^{-1}s)^*, & \text{if } t \ll s. \end{cases} \quad \forall s, t \in P.$

(4) $m(s)m(t) = \begin{cases} m(s), & \text{if } t \ll s, \\ m(t), & \text{if } s \ll t. \end{cases} \quad m(s) = m(t) \Leftrightarrow s \sim t, \forall s, t \in P.$

Proposition 3.2. (1) Let $\mathcal{A} = \text{sp}\{w(s)w(t)^* \mid s, t \in P\}$, then $\mathcal{A}$ is a dense unital $*$ -subalgebra of $W(G, P)$.

(2) The operators $\{w(s)w(t)^* \mid s, t \in P\}$ are linear independent in the sense that, if $\sum_j \lambda_j w(s_j)w(t_j)^* = 0$ with $w(s_{j_1})w(t_{j_1})^* \neq w(s_{j_2})w(t_{j_2})^*$ when $j_1 \neq j_2$, then $\lambda_j = 0$ for all j .

Proof. (1) It suffices to show for any $s_1, s_2, t_1, t_2 \in P$, $w(s_1)w(t_1)^*w(s_2)w(t_2)^* \in \mathcal{A}$, and this is clear by Proposition 3.1.

(2) If $T = \sum_j \lambda_j w(s_j)w(t_j)^* = 0$ with $w(s_{j_1})w(t_{j_1})^* \neq w(s_{j_2})w(t_{j_2})^*$ whenever $j_1 \neq j_2$, we show $\lambda_j = 0$ for all j . Without loss of generality, we may assume $t_1 \ll t_2 \ll \dots \ll t_n$.

Suppose $t_1 \sim t_2 \sim \dots \sim t_{k_1} < t_{k_1+1}$, then by $T\delta_{t_1} = 0$, we know that

$$\sum_{j=1}^{k_1} \lambda_j \delta_{s_j t_j^{-1} t_1} = 0.$$

If $\lambda_1 \neq 0$, then there exists $j \in \{2, 3, \dots, k_1\}$ such that $s_j t_j^{-1} t_1 = s_1$, so $s_j t_j^{-1} = s_1 t_1^{-1}$, which is a contradiction since $w(s_j)w(t_j)^* \neq w(s_1)w(t_1)^*$. Similarly, we have $t_2, t_3, \dots, t_{k_1} = 0$. Continue the above process, eventually we have $\lambda_j = 0$ for all j . $\square$

Denote by $\mathcal{D} = \text{clos sp}\{m(t) \mid t \in P\}$. Clearly it is a unital abelian C^* -subalgebra of $W(G, P)$. Denote by $\mathcal{D}_0$ the C^* -subalgebra of $B(\ell^2(P))$ consisting of all the operators having diagonal matrix relative to the canonical basis. It is well-known that there exists a linear and contractive map $E_0 : B(\ell^2(P)) \rightarrow \mathcal{D}_0$ determined by the following rule: the matrix of $E_0(T)$ (relative to the canonical basis) is obtained from the one of T by replacing with zero all the entries which are not situated on the principal diagonal.

Proposition 3.3 (cf. [5], Section 3.3 and Section 3.6). (1) $\mathcal{D} = \{T \in W(G, P) \mid T \text{ has diagonal matrix relative to the canonical basis of } \ell^2(P)\}$.

(2) Let $E = E_0|_{W(G, P)}$, then E is a faithful bounded linear map from $W(G, P)$ to $\mathcal{D}$ such that for any s, t in P :

$$E(w(s)w(t)^*) = \begin{cases} w(s)w(t)^*, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases}$$

Proposition 3.4. Let $\{L(t) \mid t \in P\}$ be a family of non-zero projections of the unital C^* -algebra $\mathcal{B}$ with $L(e) = 1$. Then there exists a C^* -algebra morphism $\rho :$

$\mathcal{D} \rightarrow \mathcal{B}$ such that $\rho(m(t)) = L(t), \forall t \in P$, if and only if for any $x, y, s, t \in P$:

$$(*) \quad L(x) = L(y) \text{ whenever } x \sim y, \text{ and } L(s)L(t) = \begin{cases} L(s), & \text{if } t \ll s, \\ L(t), & \text{if } s \ll t. \end{cases}$$

For L as above, ρ is faithful if and only if, $L(s) = L(t) \Leftrightarrow s \sim t$ for any $s, t \in P$.

Proof. “ $\Rightarrow$ ” clearly follows from Proposition 3.1. To prove “ $\Leftarrow$ ” it suffices to show that, for any $t_1, t_2, \dots, t_n \in P$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in C$:

$$\left\| \sum_{j=1}^n \lambda_j L(t_j) \right\| \leq \left\| \sum_{j=1}^n \lambda_j m(t_j) \right\|.$$

Without loss of generality, we may assume $t_1 \ll t_2 \ll \dots \ll t_n$, more explicitly,

$$t_1 \sim \dots \sim t_{k_1} < t_{k_1+1} \sim \dots \sim t_{k_2} < \dots < t_{k_{m-1}+1} \sim \dots \sim t_{k_m} = t_n.$$

Let $s_i = \lambda_1 + \dots + \lambda_{k_i}$ for $i = 1, \dots, m$, and $L_{i,i+1} = L(t_{k_i}) - L(t_{k_{i+1}})$ for $i = 1, \dots, m-1$. Since $T = \sum_{j=1}^n \lambda_j m(t_j)$ has diagonal matrix,

$$\begin{aligned} \left\| \sum_{j=1}^n \lambda_j m(t_j) \right\| &= \sup_{a \in P} |\langle T \delta_a, \delta_a \rangle| = \sup_{a \in P} \left| \sum_j \lambda_j \langle m(t_j) \delta_a, \delta_a \rangle \right| \\ &= \max\{|s_1|, |s_2|, \dots, |s_m|\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{j=1}^n \lambda_j L(t_j) &= (\lambda_1 + \dots + \lambda_{k_1})L(t_{k_1}) + \dots + (\lambda_{k_{m-1}+1} + \dots + \lambda_{k_m})L(t_{k_m}) \\ &= s_1 L_{1,2} + s_2 L_{2,3} + \dots + s_{m-1} L_{m-1,m} + s_m L(t_{k_m}). \end{aligned}$$

$$\text{So } \left\| \sum_{j=1}^n \lambda_j L(t_j) \right\| = \max\left\{ \max_{1 \leq i \leq m-1} \{|s_i| |L(t_{k_i}) - L(t_{k_{i+1}})|\}, |s_m| \right\}.$$

Thus we obtain

$$\left\| \sum_{j=1}^n \lambda_j L(t_j) \right\| \leq \left\| \sum_{j=1}^n \lambda_j m(t_j) \right\|.$$

The proof above shows also that, for L as above, ρ is faithful if and only if $L(s) = L(t) \Leftrightarrow s \sim t$ for any $s, t \in P$. $\square$

Remark. Let V be a representation by isometries of P on some Hilbert space H , i.e.

$$\begin{aligned} V(t)^* V(t) &= 1, \forall t \in P; V(s)V(t) = V(st), \forall s, t \in P; \\ V(e) &= 1; V(s)V(s)^* = 1, \forall s \in G_+^0, \end{aligned}$$

then V is always *covariant* in the sense that the condition $(*)$ stated in Proposition 3.4 holds with $L(t) = V(t)V(t)^*, \forall t \in P$, and if $w(s_1)w(t_1)^* = w(s_2)w(t_2)^*$ for any $s_1, s_2, t_1, t_2 \in P$, then $V(s_1)V(t_1)^* = V(s_2)V(t_2)^*$. In fact, by Proposition 3.1, we know that $t_1 \sim t_2$ and $s_1 t_1^{-1} = s_2 t_2^{-1}$. Let $a = t_1^{-1} t_2 \in G_+^0$, then

$$\begin{aligned} V(s_2)V(t_2)^* &= V(s_1 a)V(t_1 a)^* = V(s_1)V(a)(V(t_1)V(a))^* \\ &= V(s_1)V(a)V(a)^*V(t_1)^* = V(s_1)V(t_1)^*. \end{aligned}$$

It follows by Proposition 3.2 that there is a unique linear operator $\pi_V : \mathcal{A} \rightarrow \mathbb{B}(H)$ such that

$$\pi_V(w(s)w(t)^*) = V(s)V(t)^* \text{ for all } s, t \in P,$$

which directly shows that for all $s_1, s_2, t_1, t_2 \in P$:

$$\pi_V(w(s_1)w(t_1)^*w(s_2)w(t_2)^*) = \pi_V(w(s_1)w(t_1)^*)\pi_V(w(s_2)w(t_2)^*).$$

So π_V is actually a $*$ -representation.

For any $T \in \mathcal{A}$, let $T = \sum_j \lambda_j w(s_j)w(t_j)^*$, define

$$\|T\| = \sup\{\|\pi(T)\| \mid \pi \text{ is a unital } *-representation \text{ of } \mathcal{A}\}.$$

$\|T\|$ is finite and actually not greater than $\sum_j |\lambda_j|$, because each $w(s_j)w(t_j)^*$ is a partial isometry. On the other hand, the canonical identification $id : \mathcal{A} \rightarrow \mathcal{A} \subseteq W(G, P)$ gives an injective unital $*$ -representation, hence $\|T\| > 0$ if $T \neq 0$. It follows immediately that $\|\cdot\|$ is a C^* -norm on $\mathcal{A}$.

Definition. The completion of $\mathcal{A}$ with respect to $\|\cdot\|$ will be denoted by $C^*(G, P)$ and will be called the universal C^* -algebra of (G, P) . (G, P) is said to be *amenable* if $\pi_{id} : C^*(G, P) \rightarrow W(G, P)$ is one-to-one.

Remark. (G, P) is amenable if and only if, for every isometric representation $V : P \rightarrow \mathbb{B}(H)$, there is a C^* -algebra morphism $\pi_V : W(G, P) \rightarrow \mathbb{B}(H)$ such that $\pi_V(w(t)) = V(t)$ for all $t \in P$.

The proof of the following theorem is essentially the same as that of [5], the details can be found in ([5], Section 4.3, Section 4.4 and Section 4.5).

Theorem 3.5. *If G is amenable, then (G, P) is amenable.*

Now suppose G is a discrete abelian group and (G, G_+) is an ordered group. Let $P = G_F$ and define a unitary $U : \ell^2(G) \rightarrow L^2(\widehat{G})$ by $U\delta_g = \varepsilon_g$ for $g \in G$, then $U^* \circ T_{\varepsilon_g}^{G_+} \circ U = w(g)$ for all $g \in G_+$, so the two Toeplitz C^* -algebras $\mathcal{T}_r^{G_+}$ and $W(G, G_+)$ are unitarily equivalent. Since in this case G is amenable, by Theorem 3.5 we know that the Lemma 1.1 stated in Section 1 now holds.

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