SOME RAPIDLY CONVERGING SERIES FOR $\zeta(2n+1)$

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(Communicated by Hal L. Smith)

ABSTRACT. For a natural number n , the author derives several families of series representations for the Riemann Zeta function $\zeta(2n+1)$. Each of these series representing $\zeta(2n+1)$ converges remarkably rapidly with its general term having the order estimate:

$$O(k^{-2n-1} \cdot m^{-2k}) \quad (k \rightarrow \infty; \quad m = 2, 3, 4, 6).$$

Relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ are also pointed out.

1. INTRODUCTION AND PRELIMINARIES

The Riemann Zeta function $\zeta(s)$ and the (Hurwitz's) generalized Zeta function $\zeta(s, a)$, defined usually by (see, *e.g.*, Titchmarsh [26])

$$(1.1) \quad \zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \quad s \neq 1) \end{cases}$$

and

$$(1.2) \quad \zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (\Re(s) > 1; \quad a \neq 0, -1, -2, \dots),$$

so that

$$(1.3) \quad \zeta(s, 1) = \zeta(s) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1,$$

are known to be meromorphic (that is, analytic everywhere in the complex s -plane except for a simple pole at $s = 1$ with residue 1). Making use of the binomial theorem and the Pochhammer symbol $(\lambda)_n$ defined by

$$(1.4) \quad (\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \dots\}), \end{cases}$$

Received by the editors June 2, 1997.

1991 *Mathematics Subject Classification*. Primary 11M06, 11M35, 33B15; Secondary 11B68, 33E20, 40A30.

Key words and phrases. Zeta functions, binomial theorem, Pochhammer symbol, functional equation, harmonic numbers, l'Hôpital's rule, Bernoulli numbers, Euler polynomials, Euler's formula.

it is easily seen from the definition (1.2) that

$$(1.5) \quad \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s+k, a) t^k = \zeta(s, a-t) \quad (|t| < |a|),$$

which immediately yields the familiar identities (*cf.*, *e.g.*, Hansen [16, p. 359] where other references are also cited):

$$(1.6) \quad \begin{aligned} & \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \zeta(s+2k, a) t^{2k} \\ &= \frac{1}{2} [\zeta(s, a-t) + \zeta(s, a+t)] \quad (|t| < |a|) \end{aligned}$$

and

$$(1.7) \quad \begin{aligned} & \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s+2k+1, a) t^{2k+1} \\ &= \frac{1}{2} [\zeta(s, a-t) - \zeta(s, a+t)] \quad (|t| < |a|) \end{aligned}$$

or, equivalently,

$$(1.8) \quad \begin{aligned} & \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k+1)!} \zeta(s+2k, a) t^{2k+1} \\ &= \frac{1}{2(s-1)} [\zeta(s-1, a-t) - \zeta(s-1, a+t)] \quad (|t| < |a|). \end{aligned}$$

Since

$$(1.9) \quad \zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \quad (m \in \mathbb{N} \setminus \{1\}),$$

which follows readily from the definitions (1.1) and (1.2), the special case of the identity (1.6) when $a = 1$ and $t = 1/m$ can be rewritten in the form:

$$(1.10) \quad \begin{aligned} & \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{m^{2k}} \\ &= \begin{cases} (2^s - 1) \zeta(s) - 2^{s-1} & (m = 2), \\ \frac{1}{2} \left[(m^s - 1) \zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta\left(s, \frac{j}{m}\right) \right] & (m \in \mathbb{N} \setminus \{1, 2\}), \end{cases} \end{aligned}$$

where (and *throughout this paper*) an empty sum is to be interpreted as nil. In addition to the case $m = 2$, the formula (1.10) simplifies also in the cases when $m = 3, 4$, and 6 , and we thus obtain the identities:

$$(1.11) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{3^{2k}} = \frac{1}{2} [(3^s - 1) \zeta(s) - 3^s],$$

$$(1.12) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{4^{2k}} = \frac{1}{2} [(4^s - 2^s) \zeta(s) - 4^s],$$

and

$$(1.13) \quad \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{6^{2k}} = \frac{1}{2} [(6^s - 3^s - 2^s + 1)\zeta(s) - 6^s],$$

respectively.

Identities of this kind seem to have first appeared in the work of Ramaswami [22], who actually proved the cases $m = 2, 3$, and 6 of the general result in (1.10). Each of these three identities of Ramaswami [22] can also be found in the work of Hansen [16, p. 357], who referred to Apostol [1] as his source for the identities (1.11) and (1.13) only. As a matter of fact, Apostol [1] reproduced the identities (1.11) and (1.13) from Ramaswami's work [22] and then proved an interesting *arithmetical* generalization of these identities (see also Klusch [17, p. 520]).

In its slightly variant form:

$$(1.14) \quad \sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\zeta(s),$$

which can indeed be proven *directly* from the known special cases of (1.6) and (1.7) when $a = 1$ and $t = \frac{1}{2}$, the case $m = 2$ of the general result (1.10) was applied by Zhang and Williams [29] (and, more recently, by Cvijović and Klinowski [8]) with a view to finding two seemingly different series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$). The main object of this paper is to obtain much more rapidly converging series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) chiefly by appealing appropriately to each of the aforementioned cases ($m = 2, 3, 4$, and 6) of the general result (1.10).

The following properties of the Riemann ζ -function will be required in our investigation:

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2} \log(2\pi),$$

and (in general)

$$(1.15) \quad \begin{aligned} \zeta'(-2n) &= \lim_{\epsilon \rightarrow 0} \frac{\zeta(-2n + \epsilon)}{\epsilon} \\ &= \frac{(-1)^n}{2(2\pi)^{2n}} (2n)! \zeta(2n+1) \quad (n \in \mathbb{N}), \end{aligned}$$

where use is made of the familiar functional equation:

$$(1.16) \quad 2^s \Gamma(1-s) \zeta(1-s) \sin\left(\frac{1}{2} \pi s\right) = \pi^{1-s} \zeta(s).$$

Furthermore, by l'Hôpital's rule, we have

$$(1.17) \quad \lim_{s \rightarrow -2n} \left\{ \frac{\sin\left(\frac{1}{2} \pi s\right)}{s + 2n} \right\} = (-1)^n \frac{\pi}{2} \quad (n \in \mathbb{N})$$

and

$$(1.18) \quad \begin{aligned} \lim_{s \rightarrow -2n} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} &= \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}} (2n-2k)! \zeta(2n-2k+1) \\ &\quad (k = 1, \dots, n-1; \quad n \in \mathbb{N} \setminus \{1\}). \end{aligned}$$

2. A SET OF SERIES REPRESENTATIONS

We begin with the case $m = 2$ of the general result (1.10). Upon separating the first $n + 1$ terms of the series occurring on the left-hand side, if we transpose the terms for $k = 0$ and $k = n$ to the right-hand side, we obtain

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} + \sum_{k=n+1}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} \\ (2.1) \quad & = (2^s - 2)\zeta(s) - 2^{s-1} - \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s+2n)}{2^{2n}}, \end{aligned}$$

which readily yields the identity:

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s+2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}} \\ (2.2) \quad & = 2^{2n}(2^s - 2)\zeta(s) - 2^{s+2n-1} - \frac{(s)_{2n}}{(2n)!} \zeta(s+2n) \quad (n \in \mathbb{N}), \end{aligned}$$

it being understood, as before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (1.16) in the first term on the right-hand side of (2.2) and divide both sides by $s + 2n$. We thus find that

$$\begin{aligned} & \sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} + \sum_{k=1}^{\infty} \frac{(s)_{2n}(s+2n+1)_{2k-1}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}} \\ & = 2^{s+2n}(2^s - 2)\pi^{s-1}\Gamma(1-s)\zeta(1-s) \left\{ \frac{\sin(\frac{1}{2}\pi s)}{s+2n} \right\} \\ (2.3) \quad & - \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!} \zeta(s+2n)}{s+2n} \right\} \quad (s \neq -2n; \ n \in \mathbb{N}). \end{aligned}$$

Since

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \quad (k = 0, 1, \dots, n; \ n \in \mathbb{N}),$$

so that, obviously,

$$(2.4) \quad (-n)_n = (-1)^n n! \quad (n \in \mathbb{N}),$$

in view of the definition (1.4), it is easily seen by logarithmic differentiation that

$$(2.5) \quad \frac{d}{ds} \{(s)_n\} = (s)_n \sum_{j=0}^{n-1} \frac{1}{s+j} \quad (n \in \mathbb{N}),$$

so that

$$(2.6) \quad \left. \frac{d}{ds} \{(s)_{2n}\} \right|_{s=-2n} = -(2n)! H_{2n} \quad (n \in \mathbb{N}),$$

where H_n denotes the familiar harmonic numbers defined by

$$(2.7) \quad H_n := \sum_{j=1}^n \frac{1}{j} \quad (n \in \mathbb{N}).$$

We observe also that the limit formula (1.18) is needed in the first sum on the left-hand side of (2.3) only when this sum is nonzero (that is, only when $n \in \mathbb{N} \setminus \{1\}$). Furthermore, by l'Hôpital's rule, we have

$$(2.8) \quad \begin{aligned} & \lim_{s \rightarrow -2n} \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!} \zeta(s+2n)}{s+2n} \right\} \\ &= \left[2^{s+2n-1} \log 2 + \frac{d}{ds} \{(s)_{2n}\} \cdot \frac{\zeta(s+2n)}{(2n)!} + \frac{(s)_{2n}}{(2n)!} \zeta'(s+2n) \right] \Big|_{s=-2n} \\ &= \frac{1}{2} (H_{2n} - \log \pi) \quad (n \in \mathbb{N}). \end{aligned}$$

Finally, letting $s \rightarrow -2n$ in (2.3), and making use of the limit relationships (1.17), (1.18), and (2.8), we obtain our first series representation for $\zeta(2n+1)$:

$$(2.9) \quad \begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1} - 1} \left[\frac{H_{2n} - \log \pi}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right. \\ &\quad \left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}). \end{aligned}$$

In precisely the same manner, we can apply the identities (1.11), (1.12), and (1.13) in order to prove the following *additional* series representations for $\zeta(2n+1)$:

$$(2.10) \quad \begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1} - 1} \left[\frac{H_{2n} - \log(\frac{2}{3}\pi)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(\frac{2}{3}\pi)^{2k}} \right. \\ &\quad \left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{3^{2k}} \right] \quad (n \in \mathbb{N}); \end{aligned}$$

$$(2.11) \quad \begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[\frac{H_{2n} - \log(\frac{1}{2}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(\frac{1}{2}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N}); \end{aligned}$$

$$(2.12) \quad \begin{aligned} \zeta(2n+1) &= (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1) + 2^{2n} - 1} \left[\frac{H_{2n} - \log(\frac{1}{3}\pi)}{(2n)!} \right. \\ &\quad \left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(\frac{1}{3}\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N}). \end{aligned}$$

3. REMARKS AND OBSERVATIONS

Our series representation (2.9) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [29, p. 1591, Equation (3.16)] and (more recently) by Cvijović and Klinowski [8, p. 1265, Theorem A]. Since $\zeta(2k) \rightarrow 1$ as $k \rightarrow \infty$, the general term in our series representation (2.9) has the order estimate:

$$O(2^{-2k} \cdot k^{-2n-1}) \quad (k \rightarrow \infty; \ n \in \mathbb{N}),$$

whereas the general term in each of these earlier series representations has the order estimate:

$$O(2^{-2k} \cdot k^{-2n}) \quad (k \rightarrow \infty; \ n \in \mathbb{N}).$$

By suitably combining (2.9) and (2.11), it is fairly straightforward to obtain the series representation:

$$\begin{aligned} \zeta(2n+1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} \right. \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & \left. - 2 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}). \end{aligned} \quad (3.1)$$

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi) \quad (3.2)$$

and

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \quad (|z| < \pi), \quad (3.3)$$

respectively, it is known that (cf., e.g., Magnus *et al.* [20, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2(1-2^{n+1})}{n+1} B_{n+1} \quad (n \in \mathbb{N}) \quad (3.4)$$

and [20, p. 19]

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N}), \quad (3.5)$$

which, together, imply that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n-1)! (2^{2n}-1) \zeta(2n) \quad (n \in \mathbb{N}). \quad (3.6)$$

Making use of this last relationship (3.6), the series representation (3.1) can immediately be put in the form:

$$(3.7) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[\frac{\log 2}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n+2k)!} \left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0) \right] \quad (n \in \mathbb{N}),$$

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [27, p. 383, Theorem B].

Another interesting combination of our series representations (2.9) and (2.11) leads us to the following variant of Tsumura's result (3.1) or (3.7):

$$(3.8) \quad \zeta(2n+1) = (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1}-1} \left[\frac{H_{2n} - \log\left(\frac{1}{4}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} - 4 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k-1}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}),$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived recently by Ewell [12, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of our series representations (2.9) to (2.12) would yield the following (presumably new) companions of Ewell's result (3.8):

$$(3.9) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n+1}-1)(3^{2n}+1)} \left[\frac{H_{2n} - \log\left(\frac{1}{6}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} - 4 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k-1}-1)}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});$$

$$(3.10) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}+1)(3^{2n+1}-1)} \left[\frac{2H_{2n} - \log\left(\frac{\pi^2}{27}\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} - 6 \sum_{k=1}^{\infty} \frac{(2k-1)! (3^{2k-1}-1)}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});$$

$$\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+2} - 2^{2n+3} + 1} \left[\frac{H_{2n} - \log\left(\frac{8\pi}{27}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
& \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)! (3^{2k-1} - 2^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});
\end{aligned}
\tag{3.11}$$

$$\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+3} + 2^{2n+2} - 3^{2n+2} - 1} \left[\frac{H_{2n} - \log\left(\frac{27\pi}{128}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (4^{2k+1} - 3^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
& \left. - 24 \sum_{k=1}^{\infty} \frac{(2k-1)! (4^{2k-1} - 3^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{12^{2k}} \right] \quad (n \in \mathbb{N}),
\end{aligned}
\tag{3.12}$$

and

$$\begin{aligned}
\zeta(2n+1) = & (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n}+1) - 2^{4n+2} + 2^{2n} - 1} \left[\frac{H_{2n} - \log\left(\frac{4\pi}{27}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\
& \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)! (3^{2k-1} - 2^{2k-1})}{(2n+2k)!} \frac{\zeta(2k)}{12^{2k}} \right] \quad (n \in \mathbb{N}).
\end{aligned}
\tag{3.13}$$

Next we turn to the identity (1.7). By setting $t = 1/m$ and differentiating both sides with respect to s , we find from (1.7) that

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! m^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
& = \frac{m}{2} \frac{d}{ds} \left\{ \zeta\left(s, a - \frac{1}{m}\right) - \zeta\left(s, a + \frac{1}{m}\right) \right\} \quad (m \in \mathbb{N} \setminus \{1\}),
\end{aligned}
\tag{3.14}$$

where we have made use of the derivative formula (2.5). In particular, when $m = 2$, (3.14) immediately yields

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! 2^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
& = - \left(a - \frac{1}{2}\right)^{-s} \log\left(a - \frac{1}{2}\right).
\end{aligned}
\tag{3.15}$$

By letting $s \rightarrow -2n - 1$ ($n \in \mathbb{N}$) in the further special of this last identity (3.15) when $a = 1$, Wilton [28, p. 92] obtained the following series representation for

$\zeta(2n+1)$ (see also Hansen [16, p. 357, Entry (54.6.9)]):

$$(3.16) \quad \zeta(2n+1) = (-1)^{n-1} \pi^{2n} \left[\frac{H_{2n+1} - \log \pi}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right. \\ \left. + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which may be compared with our first series representation (2.9). As a matter of fact, since

$$(3.17) \quad \frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n \frac{(2k-1)!}{(2n+2k)!} \quad (n, k \in \mathbb{N}),$$

it is not difficult to deduce from (2.9) and (3.16) (with n replaced by $n-1$) that

$$(3.18) \quad \zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1}-1)} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k)}{\pi^{2k}} \right. \\ \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which is precisely the aforementioned *main* result of Cvijović and Klinowski [8, p. 1265, Theorem A] (see also Zhang and Williams [29, p. 1591, Equation (3.16)] where an obviously more complicated version of (3.18) was proven by applying the same identity (1.14) above).

Observing also that

$$(3.19) \quad \frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1) \frac{(2k-1)!}{(2n+2k+1)!} \quad (n, k \in \mathbb{N}),$$

we obtain yet another series representation for $\zeta(2n+1)$ by applying (2.9) and (3.16):

$$(3.20) \quad \zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n-1)2^{2n}+1} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right. \\ \left. + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \right] \quad (n \in \mathbb{N}),$$

which provides a significantly simpler (and much more rapidly convergent) version of the other *main* result of Cvijović and Klinowski [8, p. 1265, Theorem B]:

$$(3.21) \quad \zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}),$$

where the coefficients $\Omega_{n,k}$ are given explicitly by

$$(3.22) \quad \Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j+2k+1)(j+1)2^j} \quad (n \in \mathbb{N}; \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

in terms of the Bernoulli numbers defined by (3.2). Since [20, pp. 27 and 28]

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}),$$

the definition (3.22) can be rewritten at once in the form:

(3.23)

$$\Omega_{n,k} = \sum_{j=0}^n \binom{2n}{2j} \frac{B_{2n-2j}}{(2j+2k+1)(2j+1)2^{2j}} - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; \quad k \in \mathbb{N}_0),$$

or, equivalently,

$$\begin{aligned} \Omega_{n,k} = & (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=0}^n \frac{(-\pi^2)^j \zeta(2n-2j)}{(2j+1)!(2j+2k+1)} \\ & - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; \quad k \in \mathbb{N}_0), \end{aligned} \quad (3.24)$$

by virtue of the relationship (3.5). Combining the partial fractions occurring in (3.23) or (3.24), it is easily seen that

$$\begin{aligned} \Omega_{n,k} = & \frac{\prod_{\ell=0}^n \{(2k+2\ell+1)^{-1}\}}{(2n+2k)2^{2n}} \left[\sum_{j=0}^n \binom{2n}{2j} \frac{2n+2k}{2j+1} 2^{2n-2j} B_{2n-2j} \right. \\ & \cdot \left. \prod_{\substack{\ell=0 \\ (\ell \neq j)}}^n (2k+2\ell+1) - \prod_{\ell=0}^n (2k+2\ell+1) \right] \quad (n \in \mathbb{N}; \quad k \in \mathbb{N}_0). \end{aligned} \quad (3.25)$$

In view of the identity:

$$\sum_{j=0}^n \binom{2n}{2j} \frac{2^{2n-2j} B_{2n-2j}}{2j+1} = 1 = \sum_{j=0}^n \binom{2n}{2j} \frac{2^{2j} B_{2j}}{2n-2j+1}, \quad (3.26)$$

which is due essentially to Euler (*cf.*, *e.g.*, Riordan [23, p. 123, Problem 12]), the expression inside brackets in (3.25) is a polynomial in k of degree n (not $n+1$), and therefore

$$\Omega_{n,k} = O(k^{-2}) \quad (k \rightarrow \infty; \quad n \in \mathbb{N}). \quad (3.27)$$

It follows from (3.27) that the general term in (3.21) has the order estimate:

$$O(2^{-2k} \cdot k^{-2}) \quad (k \rightarrow \infty), \quad (3.28)$$

whereas the general term in our series representation (3.20) has precisely the same order estimate:

$$O(2^{-2k} \cdot k^{-2n-1}) \quad (k \rightarrow \infty; \quad n \in \mathbb{N}), \quad (3.29)$$

as that in (2.9). Thus, even in the special case when $n=1$, the series representing $\zeta(3)$ converges faster in (3.20) than in (3.21).

Various known series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) of other types include those given (for example) by Ramanujan [21] (see also Berndt [3]), Glaisher [13] (see also Hansen [16, p. 359]), Koshliakov [18], Leshchiner [19], Grosswald ([14] and [15]), Terras [25], Cohen [7], Butzer *et al.* ([5] and [6]), Dąbrowski [9], and others (see, *e.g.*, Berndt [4, pp. 275 and 276]).

We conclude this paper by remarking that a particular case of our series representation (2.12) when $n=1$ was proven, by an entirely different method, by Zhang

and Williams [30, p. 707, Theorem 9]. Furthermore, the following particular case of (3.18) when $n = 1$:

$$(3.30) \quad \zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}},$$

which is contained in a 1772 paper entitled *Exercitationes Analyticae* by Euler (see, e.g., Ayoub [2, pp. 1084–1085]), was rediscovered by Ramaswami [22] and (more recently) by Ewell [10]. In fact, Euler's formula (3.30) was reproduced by Srivastava [24, p. 7, Equation (2.23)] from the work of Ramaswami [22]. In the *current* mathematical literature, however, Euler's formula (3.30) is being attributed to Ewell [10].

ACKNOWLEDGMENTS

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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