CLASSIFICATION OF EXTREMAL CONTRACTIONS FROM SMOOTH FOURFOLDS OF $(3, 1)$ -TYPE

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ABSTRACT. We investigate divisorial contractions of extremal rays from smooth fourfolds. When the exceptional divisor is contracted to a curve, we prove that the divisor is a $\mathbb{P}^2$ -bundle or quadric bundle over a smooth curve and the contraction is the blowing up along the curve. Furthermore we determine the local analytic structure of the contraction.

0. INTRODUCTION

Let X be a projective normal variety with only terminal singularities and Y a projective normal variety.

We say that $f : X \rightarrow Y$ is a contraction of an extremal ray if

- (1) f has only connected fibers;
- (2) $-K_X$ is f -ample;
- (3) for any two f -exceptional curves C_1 and C_2 , there exists a positive rational number r such that for any $\mathbb{Q}$ -Cartier divisor D , $(D.C_1) = r(D.C_2)$.

We call the pull back of an ample divisor on Y a good supporting divisor. Furthermore we say that such a contraction is of (m, n) -type if the dimension of the exceptional locus is equal to m and the dimension of the image of the exceptional locus is equal to n . We also say that a contraction is divisorial if its exceptional locus is a divisor. (For a more general setting, see [KMM].)

When $\dim X = 4$ and X is smooth, many authors investigated contractions of extremal rays. Here are the references for them:

In [A], T. Ando determined the contractions of $(4, 3)$ -type and $(3, 2)$ -type in the case when all nontrivial fibers are 1-dimensional and also treated the contractions of $(3, 0)$ -type and $(3, 1)$ -type. (M. Beltrametti obtained similar results in [Be1].) In [A-W2], M. Andreatta and J. Wiśniewski determined isolated 2-dimensional fibers of the contractions of $(3, 2)$ -type. See also [A-W3] for the most recent developments in the study of higher dimensional contractions. In [Be2], M. Beltrametti listed the possibilities of the exceptional divisors of the contractions of $(3, 0)$ -type and in [F1], T. Fujita excluded many nonnormal possibilities in the list. In [Kac], Y. Kachi described isolated 2-dimensional fibers of the contraction of $(4, 3)$ -type. In [Kaw2], Y. Kawamata determined the contractions of $(2, 0)$ -type.

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In this paper we investigate the contractions of (3, 1)-type.

Main Theorem. *Let X be a smooth 4-fold and let $f : X \rightarrow Y$ be a contraction of an extremal ray of (3, 1)-type. Let E be the exceptional divisor of f and $C := f(E)$. Then*

- (1) C is a smooth curve;
- (2) $f|_E : E \rightarrow C$ is a $\mathbb{P}^2$ -bundle or a quadric bundle (see Definition 1.4 below) over C without multiple fibers;
- (3) f is the blowing up of Y along C ;
- (4) (Local analytic description of singularity of Y along C)
- (4p1) if $f|_E : E \rightarrow C$ is a $\mathbb{P}^2$ -bundle and $\mathcal{O}_X(-E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ for any fiber F of $f|_E$, then Y is smooth along C ;
- (4p2) if $f|_E : E \rightarrow C$ is a $\mathbb{P}^2$ -bundle and $\mathcal{O}_X(-E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of $f|_E$, then $(C \subset Y)$ is locally analytically isomorphic to $((x = y = z = 0) \subset (\mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)) \times \mathbb{C})$, where x, y, z are coordinates of $\mathbb{C}^3$ and $\mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)$ means the quotient space of $\mathbb{C}^3$ by the action of $\mathbb{Z}_2$ which acts as $(x, y, z) \rightarrow (-x, -y, -z)$;
- (4q) if $f|_E : E \rightarrow C$ is a quadric bundle, then Y is analytically locally a hypersurface $f(x, y, z, w, t) = 0$ in $\mathbb{C}^5$ with coordinates x, y, z, w, t and C is a local complete intersection $(x = y = z = w = 0)$. Let F' be the fiber of $f|_E$ over the origin and F a general fiber near F' ;
- (4q1) if F and F' are $\mathbb{P}^1 \times \mathbb{P}^1$, then we can write that

$$f(x, y, z, w, t) = x^2 + y^2 + z^2 + w^2;$$

- (4q2) if F is $\mathbb{P}^1 \times \mathbb{P}^1$ and F' is $\mathbb{F}_{2,0}$, then we can write that

$$f(x, y, z, w, t) = x^2 + y^2 + z^2 + t^m w^2 + w^3 \text{ for } m \geq 2,$$

or

$$f(x, y, z, w, t) = x^2 + y^2 + z^2 + t w^2 + g(w, t) \text{ with } g \in (w^3);$$

- (4q3) if F is $\mathbb{P}^1 \times \mathbb{P}^1$ and F' is the union of two distinct planes in $\mathbb{P}^3$, then we can write that

$$\begin{aligned} f(x, y, z, w, t) &= x^2 + y^2 + t^m z^2 + z^3 + t w^2 + g(t, z, w) \\ &\text{with } m \geq 2 \text{ and } g \in (z^2 w, z w^2, w^3), \end{aligned}$$

or

$$\begin{aligned} f(x, y, z, w, t) &= x^2 + y^2 + t z^2 + t w^2 + g(t, z, w) \text{ with } g \in (z, w)^3 \text{ and} \\ &g_3(z, w) \text{ cannot be divided by } z + \sqrt{-1}w \text{ or } z - \sqrt{-1}w, \\ &\text{where } g_3 \text{ is the homogeneous part of } g \text{ of degree 3 in } z, w; \end{aligned}$$

- (4q4) if F and F' are $\mathbb{F}_{2,0}$, then we can write that

$$f(x, y, z, w, t) = x^2 + y^2 + z^2 + w^3;$$

- (4q5) if F is $\mathbb{F}_{2,0}$ and F' is the union of two planes in $\mathbb{P}^3$, then we can write that

$$f(x, y, z, w, t) = x^2 + y^2 + t z^2 + w^3 + g(t, z, w) \text{ with } g(t, z, w) \in (z^3, z^2 w, z w^2).$$

Notation and conventions. (1) In this paper, we will work over $\mathbb{C}$, the complex number field.

(2) The $\mathbb{P}^1$ -bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ over $\mathbb{P}^1$, a Hirzebruch surface of degree a , is denoted by $\mathbb{F}_a$. Its unique negative section is denoted by C_0 and a ruling is denoted by f . The projective cone obtained from $\mathbb{F}_a$ by the contraction of C_0 is denoted by $\mathbb{F}_{a,0}$. A generating line on $\mathbb{F}_{a,0}$ is denoted by l .

1. PRELIMINARIES

Theorem 1.1. *Let X , Y , E and C be as in the main theorem. Let F be a general fiber of $f|_E : E \rightarrow C$. Then $(F, -K_X|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)), (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ or $(\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{F}_{2,0}}(1)|_{\mathbb{F}_{2,0}})$.*

Proof. In [A, Theorem 2.1], T. Ando stated this theorem but his proof of irreducibility of F is insufficient. Since the image of E by $H|_E$, where H is a good supporting divisor, is 1-dimensional, Bertini's theorem does not derive the irreducibility of F . Since we don't know that E is normal a priori, irreducibility of F is subtle. So we will prove it here. Once we prove it, the results follow from [A, Theorem 2.1]. (M. Beltrametti stated a slightly weaker result in [Be1, Proposition 2.7].) We assume that F is reducible and obtain a contradiction. Let H be a good supporting divisor of f . We may assume that H is a smooth variety and locally $F = H \cap E$. By the adjunction formula, $-K_F = -K_X - E - H|_F$, but since $H|_F \sim 0$, we have $-K_F = -K_X - E|_F$. Note that $-K_X|_F$ and $-E|_F$ are ample Cartier divisors on F . In particular $-K_F$ is ample on F and hence F is a generalized del Pezzo surface (i.e., a Gorenstein (possibly reducible) anticanonically polarized surface). Write $F = \bigcup F_i$, where F_i is an irreducible component of F . By [R3, 1.1], $(F_i, -K_F|_{F_i})$ is one of the following:

- (a) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(i))$, where i is 1 or 2;
- (b) $(\mathbb{F}_{a,0}, \mathcal{O}_{\mathbb{F}_{a,0}}(al))$;
- (c) $(\mathbb{F}_a, \mathcal{O}_{\mathbb{F}_a}(C_0 + (a+1)f))$;
- (d) $(\mathbb{F}_a, \mathcal{O}_{\mathbb{F}_a}(C_0 + (a+2)f))$.

But since $-K_F$ is a sum of two ample Cartier divisors, (a) with $i = 1$, (b), (c) and (d) are impossible. So we obtain that $(F_i, -K_F|_{F_i}) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $-K_X|_{F_i} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. Furthermore by [R3, 1.3, 1.4 and 1.5], F is a union of two $\mathbb{P}^2$'s which intersect along a line on them. But this is impossible since $N_{F_i/H} \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ and hence there is a bimeromorphic contraction which contracts only one F_i . $\square$

The next theorem about freeness is very useful for the classification of low dimensional fibers of an extremal contraction.

Theorem 1.2 (M. Andreatta and J. Wiśniewski). *Let X and Y be normal log terminal varieties and $f : X \rightarrow Y$ a projective morphism. Let L be an f -ample line bundle on X and F a fiber of f . Assume that f is the adjoint contraction supported by $K_X + rL$ and either $\dim F < r + 1$ if $\dim Y < \dim X$ or $\dim F \leq r + 1$ if $\dim Y = \dim X$.*

Then $f^*f_*L \rightarrow L$ is surjective at every point of F , i.e., L is f -free at every point of F .

Proof. See [A-W1, Theorem 3.1]. Andreatta and Wiśniewski assume that L is ample but their proof works also for the case that L is relatively ample. $\square$

Theorem 1.3 (P. M. H. Wilson). *Let X_0 be a smooth 3-fold and Y_0 a 3-fold with only canonical singularities. Let $f : X_0 \rightarrow Y_0$ be a crepant birational (i.e., f satisfies $K_{X_0} = f^*K_{Y_0}$) projective morphism which contracts an irreducible divisor*

to a curve. Let E_0 be the exceptional divisor and $C := f(E_0)$. Assume that $-E_0$ is f -ample. Then C is a smooth curve and Y_0 has only cDV singularity along C .

Proof. See [W1, Theorem 2.2] and [W2, Proposition 3.1]. Wilson assumes that X_0 is a Calabi-Yau manifold and f is primitive. But by the following modification, his proof works in the local situation and under the assumption that $-E_0$ is f -ample instead of the assumption that f is primitive. In Wilson's proof [W1, the 6th line of page 569], he claimed that f coincides with the blowing up of Y_0 along C using primitivity of f . We replace the argument as follows:

Let $f' : X'_0 \rightarrow Y_0$ be the blowing up of Y_0 along C and E'_0 the exceptional divisor. By the universal property of blowing up, $g : X_0 \rightarrow X'_0$ is naturally defined. We know that $g(E_0) = E'_0$ since E_0 is irreducible. Note that E'_0 is a Cartier divisor by the property of blowing up. If g is not an isomorphism, it contracts finitely many curves since E_0 is irreducible. But it is impossible since E'_0 is Cartier and $-E_0$ is positive for the contracted curves by the assumption that $-E_0$ is f -ample. $\square$

Definition 1.4. Let E be a normal projective 3-fold and C a smooth curve. Let $f : E \rightarrow C$ be a projective surjective morphism. We say that $f : E \rightarrow C$ is a quadric bundle if the following conditions are satisfied:

- (1) There exists a f -very ample line bundle L on E and f_*L is locally free of rank 4;
- (2) L gives the embedding $E \rightarrow \mathbb{P}(f_*L)$ over C ;
- (3) For any closed point s of C , E_s (the fiber of f over s) is a quadric in $\mathbb{P}^3 \simeq \mathbb{P}(f_*L)_s$ (the fiber of $\mathbb{P}(f_*L) \rightarrow C$ over s).

If we can take such an L , we say that $f : E \rightarrow C$ is the quadric bundle associated to L .

Remark 1.5. We require that E is normal in Definition 1.4 so general fibers of f are irreducible.

2. PROOF OF THE MAIN THEOREM

Proof of (1). Since $L := -K_X$ is f -ample, we can apply Theorem 1.2 for this f and L with $r = 1$. (Remark that the dimension of fibers of f is 2.) Then for any point P of C and a suitable affine neighborhood U of P in Y , $|L|_{f^{-1}(U)}$ has no base points on $f^{-1}(U)$. Let X_0 be a general smooth member of $|L|_{f^{-1}(U)}$, $Y_0 := f(X_0)$ and $E_0 := E|_{X_0}$. Then $f|_{X_0}$ is crepant, E_0 is irreducible (by Bertini's theorem) and $-E_0$ is $f|_{X_0}$ -ample. From this, we can apply Theorem 1.3 for X_0 , Y_0 and E_0 . Hence C is smooth. $\square$

Proof of (2). Let F be a general fiber of f .

If $(F, -K_X|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, let L' be $\mathcal{O}_E(-K_X)$.

If $(F, -K_X|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ or $(\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$, let L' be $\mathcal{O}_E(-E)$. Then we will prove that

- (i) if $F \simeq \mathbb{P}^2$, $f|_E : E \rightarrow C$ is a $\mathbb{P}^2$ -bundle $\mathbb{P}((f|_E)_*L')$;
- (ii) if $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_{2,0}$, $f|_E : E \rightarrow C$ is the quadric bundle associated to L' .

Claim 2.1. $(f|_E)_*L'$ is locally free.

Proof. By the proof of (1) above, C is smooth so $f|_E$ is flat. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-K_X - E) \rightarrow \mathcal{O}_X(-K_X) \rightarrow L' \rightarrow 0$$

if $L' = \mathcal{O}_E(-K_X)$ or

$$0 \rightarrow \mathcal{O}_X(-2E) \rightarrow \mathcal{O}_X(-E) \rightarrow L' \rightarrow 0$$

if $L' = \mathcal{O}_E(-E)$. By the Kodaira-Kawamata-Viehweg vanishing theorem, we have $R^i(f|_E)_*L' = 0$ for any $i > 0$. On the other hand, we have $H^i(E_s, L'_s) = 0$ for $i \gg 0$ and any $s \in C$. In particular L' is flat over C , $R^i(f|_E)_*L' \otimes k(s) = H^i(E_s, L'_s)$ for $i \gg 0$ and $R^i(f|_E)_*L'$ is locally free for any $i > 0$. So by using the cohomology and base change theorem inductively (cf. [H, III Theorem 12.11]), we know that $(f|_E)_*L'$ is locally free. $\square$

Claim 2.2. L' is $f|_E$ -free.

Proof. We consider the commutative diagram

$$(2.2.1) \quad \begin{array}{ccc} f^*f_*\mathcal{O}_X(-K_X) & \longrightarrow & (f|_E)^*(f|_E)_*L' \\ \downarrow & & \downarrow \\ \mathcal{O}_X(-K_X) & \longrightarrow & L' \end{array}$$

if $L' = \mathcal{O}_E(-K_X)$ or

$$(2.2.2) \quad \begin{array}{ccc} f^*f_*\mathcal{O}_X(-E) & \longrightarrow & (f|_E)^*(f|_E)_*L' \\ \downarrow & & \downarrow \\ \mathcal{O}_X(-E) & \longrightarrow & L' \end{array}$$

if $L' = \mathcal{O}_E(-E)$.

We see the left vertical arrow is surjective by Theorem 1.2 and so is the bottom arrow clearly. So the right vertical arrow must be surjective, i.e., L' is $f|_E$ -free. $\square$

By these, we obtain the morphism $g : E \rightarrow \mathbb{P}((f|_E)_*L')$ defined by L' .

If we are in case (i), g is birational since the general fiber of f is $\mathbb{P}^2$ and $\mathbb{P}((f|_E)_*L')$ is a $\mathbb{P}^2$ -bundle. Furthermore g is finite since L' is $f|_E$ -ample. So by Zariski's Main Theorem, g is an isomorphism and we are done.

If we are in case (ii), we will show first that f is the blowing up of Y along C .

Let $f' : X' \rightarrow Y$ be the blowing up of Y along C . By Theorem 1.2, we have the surjection $f^*f_*\mathcal{O}_X(-E) \rightarrow \mathcal{O}_X(-E)$. Since $f_*\mathcal{O}_X(-E) = \mathcal{I}_C$, where $\mathcal{I}_C$ is the ideal sheaf of C , we have the morphism $h : X \rightarrow X'$ which satisfies $f = f' \circ h$ by the universal property of blowing up. By Theorem 1.3 and the proof of the main theorem (1), near any point x on C , there is a general member of $|-K_Y|$ through x which has only cDV singularities along C . Since $|-K_Y|$ is a Cartier divisor in case (ii), Y also has hypersurface singularities along C . If we consider f' analytically locally near x , we know X' also has hypersurface singularities since Y is a hypersurface in $\mathbb{C}^5$ and C is a smooth curve. Furthermore X' is nonsingular along a general fiber of f' since f and f' coincide over a general point of C . Hence X' is normal. Since f contracts only an irreducible divisor, h contracts no divisor. If h is not finite, h is a small contraction such that for any contracted curve m , $-K_X.m < 0$. But because X' is Gorenstein, this is impossible. Hence h is finite birational, which in turn shows h is an isomorphism by Zariski's Main Theorem.

By this, E is naturally embedded in $\mathbb{P}((f|_E)_*L')$. This embedding coincides with g . We can easily check that E is the quadric bundle associated to L' .

We complete the proof of the main theorem (2) without excluding multiple fibers in case (ii). This is done in the proof of the main theorem (4). $\square$

Proof of (3). In case E is a quadric bundle, the proof was given in the proof of (2). In case E is a $\mathbb{P}^2$ -bundle and $(F, -E|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ for a fiber F of f , this is well known (see [N]). In case E is a $\mathbb{P}^2$ -bundle and $(F, -E|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for a fiber F of f , we can proceed as follows: let P be any point of C and take a general hyperplane section A through P . Let H be the pull back of A . Then H is smooth along $E|_H \simeq \mathbb{P}^2$ since $E|_H$ is smooth and a Cartier divisor of H . So $f|_H$ is a contraction of an extremal ray from a smooth 3-fold near the fiber over P . Then by Mori's classification [Mo2], A has a singularity analytically isomorphic to $o \in \mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)$ at P and $f|_H$ is the blowing up of A at P . Hence we are done. $\square$

Proof of (4). (4p1) This is well known (see [N]).

(4p2) $o \in \mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)$ is rigid by [S], so we are done.

(4q) By the proof of (2), Y is locally a hypersurface in $\mathbb{C}^5$ and C is a local complete intersection in $\mathbb{C}^5$. The rest are easy calculations, so we treat only the following for a sample.

Claim 2.5. F' is not multiple.

Proof. Assume that F' is multiple. We treat only the case F is $\mathbb{P}^1 \times \mathbb{P}^1$. By the proof of (2), we can assume that $f(x, y, z, w, t) = x^2 + t^m y^2 + t^n z^2 + t^l w^2 + g(y, z, w, t)$ with $g(y, z, w, t) \in (y, z, w)^3$. We will show that the 4-fold obtained by the blow up must be singular (we say in short that the blow up is singular). For simplicity, we assume furthermore that $m = n = l = 1$. By easy computations, the blow up is singular if and only if $x = t = y^2 + z^2 + w^2 = g_3(y, z, w, t) = 0$ has a solution. But the equation always has a solution, so we are done. $\square$

This completes the proof of the main theorem. $\square$

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