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THE NORMAL HOMOGENEOUS SPACE $(\mathsf{SU}(3)\times\mathsf{SO}(3))/\mathsf{U}^\bullet(2)$ HAS POSITIVE SECTIONAL CURVATURE

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ABSTRACT. We give a new description of the positively curved seven-dimensional Aloff-Wallach spaces M^7_{kl}. In particular, this description exhibits M^7_{11} as a normal homogeneous space, although it does not occur in Berger's classification (1961).

A theorem of Berger [1961] states that a simply connected, normal homogeneous space of positive sectional curvature is either diffeomorphic to a compact rank-one symmetric space, $\mathbb{S}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, $\mathrm{Ca}\mathbb{P}^2$, or to one of the two following exceptional spaces:

1.) $V_1 := \mathsf{Sp}(2)/\mathsf{SU}(2)$, where the Lie algebra $\mathfrak{su}(2)\subset\mathfrak{sp}(2)$ is given by

$$\mathrm{span}_{\mathbb{R}}\left\{\begin{pmatrix}\frac{3}{2}i & 0\\ 0 & \frac{1}{2}i\end{pmatrix},\begin{pmatrix}0 & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & j\end{pmatrix},\begin{pmatrix}0 & \frac{1}{2}\sqrt{3}\,i\\ \frac{1}{2}\sqrt{3}\,i & k\end{pmatrix}\right\}.$$

2.) $V_2 := \mathsf{SU}(5)/\mathsf{H}$, where the group H is given by

$$\mathsf{H} := \left\{\begin{pmatrix} zA & 0\\ 0 & \bar z^4\end{pmatrix}\;\middle|\; A\in\mathsf{Sp}(2)\subset\mathsf{SU}(4)\,,\ z\in\mathsf{S}^1\subset\mathbb{C}\right\}\subset\mathsf{U}(4)\subset\mathsf{SU}(5).$$

In particular H is isomorphic to $\big(\mathsf{Sp}(2)\times\mathsf{S}^1\big)/\{\pm(\mathrm{id},1)\}$.

Here normal homogeneous means that the metrics on V_1 and V_2 are induced from biinvariant metrics on $\mathsf{Sp}(2)$ and $\mathsf{SU}(5)$, respectively. This theorem is not correct. We will show that there is a third exception:

3.) $V_3 := \big(\mathsf{SU}(3)\times\mathsf{SO}(3)\big)/\mathsf{U}^\bullet(2)$, where $\mathsf{U}^\bullet(2)$ is the image under the embedding $(\iota,\pi)\colon \mathsf{U}(2)\hookrightarrow\mathsf{SU}(3)\times\mathsf{SO}(3)$ given by the natural inclusion

$$\iota(A) := \begin{pmatrix} A & 0\\ 0 & \det A^{-1}\end{pmatrix}\qquad \text{for } A\in\mathsf{U}(2)$$

and the projection $\pi\colon \mathsf{U}(2)\to\mathsf{U}(2)/\mathsf{S}^1\cong\mathsf{SO}(3)$; here $\mathsf{S}^1\subset\mathsf{U}(2)$ denotes the center of $\mathsf{U}(2)$. On $\mathsf{SU}(3)\times\mathsf{SO}(3)$ we consider the 1-parameter family of biinvariant metrics $\tilde g_\lambda := -\big(B_{\mathfrak{su}(3)}\times\lambda B_{\mathfrak{so}(3)}\big)$ for $\lambda>0$, where $B_{\mathfrak{su}(3)}$ and $B_{\mathfrak{so}(3)}$ are the Killing forms of $\mathfrak{su}(3)$ and $\mathfrak{so}(3)$, respectively. The induced metric on the quotient V_3 which turns the projection into a Riemannian submersion will also be denoted by $\tilde g_\lambda$.

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In fact, Aloff and Wallach [AW1972] have introduced $(V_3, \tilde{g}_\lambda)$ from a different point of view. They studied for positive integers k and l the groups

$$\mathsf{T}_{kl} := \left\{ \begin{pmatrix} z^k & & \\ & z^l & \\ & & \bar{z}^{k+l} \end{pmatrix} \,\middle|\, z \in \mathsf{S}^1 \subset \mathbb{C} \right\} \subset \mathsf{U}(2) \subset \mathsf{SU}(3),$$

and the 1-parameter family of left-invariant, $\mathrm{Ad}(\mathsf{U}(2))$-invariant metrics on $\mathsf{SU}(3)$ given by

$$g_t\big(u + x, v + y\big) := -(1 + t)\, B_{\mathfrak{su}(3)}\big(u, v\big) - B_{\mathfrak{su}(3)}\big(x, y\big)$$

where $t \in (-1, \infty)$, $u, v \in \mathfrak{u}(2)$ and $x, y \in \mathfrak{u}(2)^\perp$. Aloff and Wallach have shown that the space $\big(M^7_{kl}, g_t\big) := \big(\mathsf{SU}(3), g_t\big)/\mathsf{T}_{kl}$ has positive sectional curvature if and only if $t \in (-1, 0)$.

In the special case $k = l = 1$ we will prove

Proposition 1. $\big(V_3, \tilde{g}_\lambda\big)$ *is isometric to* $\big(M^7_{11}, g_t\big)$ *for* $t = -\frac{3}{2\lambda+3}$.

According to the proof that we give below, the isometry is obtained upon combining the canonical diffeomorphisms between the following spaces

$$\begin{aligned} \mathsf{SU}(3)/\mathsf{T}_{11} &\cong \big((\mathsf{SU}(3)/\mathsf{T}_{11}) \times \mathsf{U}(2)\big)/\mathsf{U}^\blacktriangle(2) \\ &= \big(\mathsf{SU}(3) \times (\mathsf{U}(2)/\mathsf{S}^1)\big)/\mathsf{U}^\blacktriangle(2) \\ &\cong \big(\mathsf{SU}(3) \times \mathsf{SO}(3)\big)/\mathsf{U}^\bullet(2), \end{aligned}$$

where $\mathsf{U}^\blacktriangle(2) \subset \mathsf{SU}(3) \times \mathsf{U}(2)$ denotes the image of $\mathsf{U}(2)$ under the diagonal embedding (ι, id) and $\mathsf{S}^1 = \mathsf{T}_{11}$ is the center of $\mathsf{U}(2)$.

By Proposition 1 the 1-parameter family of normal homogeneous metrics on V_3 coincides with the positively curved Aloff–Wallach metrics on M^7_{11}. Püttmann [1997] has computed the pinching constants of the metrics g_t on M^7_{11}, i.e., the ratios of the minimum and the maximum of their sectional curvatures. He arrived at the conclusion that the optimal pinching constant for these metrics is $\frac{1}{37}$ and is attained at $t = -\frac{3}{5}$. It is a curious fact that the optimal metric $g_{-3/5}$ is Einstein and, by Proposition 1, also the natural metric induced by the Killing form on the product $\mathfrak{su}(3) \times \mathfrak{so}(3)$.

The other positively curved Aloff–Wallach examples are not normal homogeneous spaces. However, they may be considered as "normal biquotients", since their metrics can be constructed from biinvariant metrics on Lie groups in the following way:

Proposition 2. *We consider the biinvariant metric* $\hat{g}_\lambda$ *on* $\mathsf{SU}(3) \times \mathsf{U}(2)$ *given by*

$$\hat{g}_\lambda\big((x, u), (y, v)\big) \quad := \quad -B_{\mathfrak{su}(3)}\big(x, y\big) - \tfrac{2}{3}\lambda\, B_{\mathfrak{su}(3)}\big(\iota_*(u), \iota_*(v)\big)$$

for $x, y \in \mathfrak{su}(3)$, $u, v \in \mathfrak{u}(2)$ *and assume* $t = -\frac{3}{2\lambda+3}$. *The map*

$$\begin{aligned} F\colon \big(\mathsf{SU}(3), g_t\big) \quad &\to \quad \big(\mathsf{SU}(3) \times \mathsf{U}(2), \hat{g}_\lambda\big)/\mathsf{U}^\blacktriangle(2), \\ A \quad &\mapsto \quad (A, e) \cdot \mathsf{U}^\blacktriangle(2) \end{aligned}$$

itself is an isometry. Furthermore, it induces an isometry between the Aloff–Wallach space (M^7_{kl}, g_t) *and the biquotient*

$$\big(Q^7_{kl}, \hat{g}_\lambda\big) := \big(\{e\} \times \mathsf{T}_{kl}\big)\backslash\big(\mathsf{SU}(3) \times \mathsf{U}(2), \hat{g}_\lambda\big)/\mathsf{U}^\blacktriangle(2).$$

The factor $\frac{2}{3}$ in the definition of $\hat{g}_\lambda$ has been chosen such that the restriction of $-\hat{g}_1$ to $\mathfrak{su}(3) \times \mathfrak{su}(2)$ coincides with the Killing form of $\mathfrak{su}(3) \times \mathfrak{su}(2)$.

Remarks. 1. Wallach [W1972] and Bérard Bergery [1976] have classified the simply connected, positively curved, homogeneous spaces in even and odd dimensions. More precisely, they listed all pairs $(\mathfrak{g},\mathfrak{h})$ of Lie algebras with $\mathfrak{h}\subset\mathfrak{g}$ that correspond to positively curved, homogeneous spaces G/H. Using this classification, it can be shown that a simply connected, normal homogeneous space of positive sectional curvature is either diffeomorphic to a compact rank-one symmetric space or to one of the exceptional spaces V_1, V_2 and V_3.

2. The pair $(A_2\oplus A_1, A_1^\bullet\oplus\mathbb{R})$ corresponding to V_3 appears in the classification of Bérard Bergery [1976]. For the geometric properties of V_3, however, he refers to an earlier paper [1975] where he claims that V_3 does not admit a normal homogeneous metric of positive sectional curvature. This remark is related to an omission in Berger's classification [1961] of positively curved, normal homogeneous spaces, which can be traced back to the treatment of the non-simple case on page 218 in his paper. There the equation "$\dim(\mathsf{T}'_{\mathbf{1}})=\dim(\mathbf{T}\cap\mathbf{L_1})+1$" does not hold for the pair $(A_2\oplus A_1, A_1^\bullet\oplus\mathbb{R})$.

3. The natural operation of $\mathsf{Sp}(2)$ on V_1 has kernel $\{\pm\,\mathrm{id}\}$. Using the fact that $\mathsf{Sp}(2)/\{\pm\,\mathrm{id}\}\cong\mathsf{SO}(5)$ it follows that V_1 can be described as quotient $\mathsf{SO}(5)/\mathsf{SO}(3)$. Eschenburg[1] has observed that the corresponding embedding $\mathsf{SO}(3)\hookrightarrow\mathsf{SO}(5)$ is induced by the canonical action of $\mathsf{SO}(3)$ on the space of traceless, symmetric, real 3×3-matrices.

4. The pinching constants of V_1 and V_2 are $\frac{1}{37}$ and $\frac{16}{29\cdot37}$, respectively (see Eliasson [1966] and Heintze [1971]). Püttmann [1997] proved that V_2 admits a homogeneous metric with pinching constant $\frac{1}{37}$, too. This metric is obtained upon shrinking the natural metric in the direction of the fibers of the canonical projection $V_2=\mathsf{SU}(5)/\mathsf{H}\to\mathsf{SU}(5)/\mathsf{U}(4)\cong\mathbb{CP}^4$. Thus V_2 with this metric can still be described as a "normal biquotient".

5. Püttmann [1997] also studied for $k\neq l$ the curvature of all homogeneous metrics on M^7_{kl} which up to scaling constitute a 3-parameter family. He has shown that in general the optimal pinching constant is not attained in the 1-parameter Aloff–Wallach family.

Proof of Proposition 1. By Proposition 2 it is sufficient to show that the manifolds $\big(Q^7_{11},\hat g_\lambda\big)$ and $\big(V_3,\tilde g_\lambda\big)$ are isometric. The subgroup T_{11} is the center of $\mathsf{U}(2)$, and $\mathsf{T}_{11}\backslash\mathsf{U}(2)=\mathsf{U}(2)/\mathsf{T}_{11}$ is the Lie group $\mathsf{SO}(3)$. Since a Lie algebra isomorphism is a linear isometry provided that both Lie algebras are equipped with their Killing forms, it follows that the homomorphism

$$\varphi\colon \big(\mathsf{SU}(3)\times\mathsf{U}(2),\hat g_\lambda\big)\to\mathsf{SU}(3)\times\big(\mathsf{U}(2)/\mathsf{T}_{11}\big)\cong\big(\mathsf{SU}(3)\times\mathsf{SO}(3),\tilde g_\lambda\big)$$

is a Riemannian submersion. Moreover φ maps the group $\big(\{e\}\times\mathsf{T}_{11}\big)\cdot\mathsf{U}^\blacktriangle(2)$ onto $\mathsf{U}^\bullet(2)$ and thus it induces an isometry $\big(Q^7_{11},\hat g_\lambda\big)\to\big(V_3,\tilde g_\lambda\big)$. $\square$

Proof of Proposition 2. Let $\sigma\colon \big(\mathsf{SU}(3)\times\mathsf{U}(2),\hat g_\lambda\big)\to\big(\mathsf{SU}(3)\times\mathsf{U}(2)\big)/\mathsf{U}^\blacktriangle(2)$ be the projection. Following our conventions we shall also write $\hat g_\lambda$ for the induced metric on the image of σ.

For $x\in\mathfrak{u}(2)^\perp$ the vector $(x,0)\in\mathfrak{su}(3)\times\mathfrak{u}(2)$ is horizontal with respect to σ and thus $\|x\|_{g_t}=\|F_*(x)\|_{\hat g_\lambda}$. For $v\in\mathfrak{u}(2)$ the horizontal component $(v,0)^h$ of $(v,0)$ in

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$\big(\mathfrak{su}(3)\times\mathfrak{u}(2),\hat g_\lambda\big)$ is given by

$$(v,0)^h=\Big(\frac{2\lambda}{2\lambda+3}\,v\,,\ \frac{-3}{2\lambda+3}\,v\Big).$$

Hence,

$$\big\|F_*(v)\big\|^2_{\hat g_\lambda}=\big\|(v,0)^h\big\|^2_{\hat g_\lambda}=-\frac{4\lambda^2+6\lambda}{(2\lambda+3)^2}\,B_{\mathfrak{su}(3)}\big(v,v\big)=\big\|v\big\|^2_{g_t}.$$

Moreover, the vectors $F_*(x)$ and $F_*(v)$ are perpendicular and thus F is an isometry.

Next we observe that

$$F(gs^{-1})=(gs^{-1},e)\cdot\mathsf{U}^{\blacktriangle}(2)=(g,s)\cdot\mathsf{U}^{\blacktriangle}(2)\qquad\text{for } s\in\mathsf{T}_{kl}.$$

In other words, F maps the left cosets of T_{kl} in $\mathsf{SU}(3)$ onto the fibers of the projection

$$\big(\mathsf{SU}(3)\times\mathsf{U}(2)\big)/\mathsf{U}^{\blacktriangle}(2)\longrightarrow\big(\{e\}\times\mathsf{T}_{kl}\big)\backslash\big(\mathsf{SU}(3)\times\mathsf{U}(2)\big)/\mathsf{U}^{\blacktriangle}(2).$$

Therefore F induces an isometry $\big(M^7_{kl},g_t\big)\to\big(Q^7_{kl},\hat g_\lambda\big)$. □

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