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C^1 APPROXIMATIONS OF INERTIAL MANIFOLDS VIA FINITE DIFFERENCES

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ABSTRACT. We construct an inertial manifold for the evolution equation as a limit of the inertial manifolds for the difference approximations of the Trotter-Kato type and show that this limit is taken in a C^1 topology.

1. INTRODUCTION

We shall study a class of nonlinear dissipative partial differential equations (PDE for short) that have inertial manifolds (IM for short). The theory of IMs allows us to reduce the long-time behavior of the PDE to that of a finite-dimensional dynamical system. In order to implement the reduced finite dynamical system computationally, one would need to know the explicit form of the IM. However, even when existence of an IM can be established, the theory does not provide it in an explicit form. Thus, a number of approximate IMs have been considered in the literature. See, e.g., [2], [3], [4], [6], [9], [10], [11], [12].

In this paper, from the point of finite differences we shall construct an IM for the PDE. Indeed, the IM is constructed as a limit of IMs for the associated finite difference equations and the limit is taken in a C^1 topology. This means that on one hand the existence of the IMs for the finite difference equations assures the existence of an IM for the PDE, and on the other the IMs for the finite differences can be viewed as a small C^1 perturbation of that for the PDE. The C^1 closeness of the IMs would be a necessary and important step toward establishing a relationship between the dynamics of the PDE and its approximation (see [8], [14]).

Each of the PDEs can be viewed as an evolution equation in a Banach space Y

$$du(t)/dt = Au(t) + Fu(t), \qquad t \in \mathbb{R}^+ \equiv [0, \infty), \tag{1.1}$$

with a closed linear operator A in Y and $F \in C^1(X, Y)$, where X is a Banach space continuously embedded in Y.

We approximate (1.1) by the following discrete scheme:

$$x_\ell^n = C(\lambda_\ell)x_\ell^{n-1} + \lambda_\ell K_\ell F_\ell(x_\ell^{n-1}), \quad n, \ell \in \mathbb{N}, \tag{1.2}$$

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in a space Y_ℓ approximating Y in some sense, where $\lambda_\ell \downarrow 0$ as $\ell \to \infty$, $C(\lambda_\ell)$ and K_ℓ are given operators in $B(Y_\ell, Y_\ell)$ and F_ℓ is a given nonlinear operator in Y_ℓ stated below. We denote by $B(W, Z)$ the space of bounded linear operators from a Banach space W into a Banach space Z. The norm in $B(W, Z)$ will be denoted by $\|\cdot\|_{W,Z}$.

2. Assumptions and result

We shall make the following assumptions.

(C1) Let X and Y be reflexive Banach spaces such that X is densely and continuously embedded in Y and such that $Y = Y_1 \oplus Y_2$, the direct sum of a finite dimensional subspace Y_1 and a closed subspace Y_2.

(C2) For each $\ell \in \mathbb{N}$ let X_ℓ and Y_ℓ be Banach spaces with norms $\|\cdot\|_\ell$ and $|\cdot|_\ell$, respectively, such that X_ℓ is continuously embedded in Y_ℓ. Moreover, there exist $V_\ell \in B(Y, Y_\ell) \cap B(X, X_\ell)$ and $W_\ell \in B(Y_\ell, Y) \cap B(X_\ell, X)$ such that $\lim_{\ell\to\infty} |V_\ell y|_\ell = |y|$, $\lim_{\ell\to\infty} \|V_\ell x\|_\ell = \|x\|$, $\lim_{\ell\to\infty} |W_\ell V_\ell y - y| = 0$ and $V_\ell W_\ell z = z$ for $x \in X, y \in Y, z \in Y_\ell$ and such that both $\|W_\ell\|_{Y_\ell,Y}$ and $\|W_\ell\|_{X_\ell,X}$ are bounded in ℓ.

(C3) There exist closed subspaces $Y_{\ell 1}$ and $Y_{\ell 2}$ such that $Y_\ell = Y_{\ell 1} \oplus Y_{\ell 2}$, $V_\ell P_i = P_{\ell i} V_\ell$ and $W_\ell P_{\ell i} = P_i W_\ell$ for $i = 1, 2$, where P_i (resp. $P_{\ell i}$) denotes a projection from Y onto Y_i (resp. Y_ℓ onto $Y_{\ell i}$).

(C4) The linear operators $C(\lambda_\ell)$ and K_ℓ satisfy: (i) there exist $M \geq 0$ and $\omega \geq 0$ such that $|C(\lambda_\ell)^n y|_\ell \leq M e^{\omega n \lambda_\ell} |y|_\ell$ and $|K_\ell y|_\ell \leq M e^{\omega \lambda_\ell} |y|_\ell$ for $\ell, n \in \mathbb{N}, y \in Y_\ell$; (ii) $\lim_{\ell\to\infty} |(K_\ell - I) V_\ell y|_\ell = 0$ for $y \in Y$; (iii) for each $\ell, \ell' \in \mathbb{N}$ and $i = 1, 2$, $C(\lambda_\ell)$ commutes with $P_{\ell i}$, $C(\lambda_\ell)$ with K_ℓ, K_ℓ with $P_{\ell i}$, $\tilde{C}(\lambda_\ell)$ with $\tilde{C}(\lambda_{\ell'})$ and $\tilde{C}(\lambda_\ell)$ with $\tilde{K}_{\ell'}$, respectively, where $\tilde{C}(\lambda_\ell) = W_\ell C(\lambda_\ell) V_\ell$ and $\tilde{K}_\ell = W_\ell K_\ell V_\ell$.

(C5) A is a densely defined linear operator in Y such that $Y_1 \subset D(A)$, the range of $I - \lambda_0 A$ is dense in Y for some $\lambda_0 > 0$ and

$$\lim_{\ell\to\infty} |\lambda_\ell^{-1}(C(\lambda_\ell) - I) V_\ell y - V_\ell A y|_\ell = 0 \quad \text{for } y \in D(A).$$

(C6) The inverse of $C(\lambda_\ell) P_{\ell 1}$ exists in $B(Y_{\ell 1})$ and there exist constants $\alpha, \beta > 0$, $\gamma \in [0, 1), \eta < -\max\{\alpha, \beta\}$ and $M_1, \cdots, M_5 \geq 0$ such that

$$\|P_{\ell 1} y\|_\ell \leq M_1 |P_{\ell 1} y|_\ell, \tag{2.1}$$

$$|[C(\lambda_\ell) P_{\ell 1}]^{-n} P_{\ell 1} y|_\ell \leq M_2 e^{-(\alpha+\eta) n \lambda_\ell} |y|_\ell, \tag{2.2}$$

$$\|C(\lambda_\ell)^n P_{\ell 2} x\|_\ell \leq M_3 e^{(\eta-\beta) n \lambda_\ell} \|x\|_\ell, \tag{2.3}$$

$$\|C(\lambda_\ell)^n P_{\ell 2} K_\ell y\|_\ell \leq \{M_4((n+1)\lambda_\ell)^{-\gamma} + M_5\} e^{(\eta-\beta) n \lambda_\ell} |y|_\ell \tag{2.4}$$
$$\text{for} \quad n \geq 0, \ell \geq 1, x \in X_\ell, y \in Y_\ell.$$

(C7) $F_\ell \in C^1(X_\ell, Y_\ell)$ and there exists a constant $L_F \geq 0$ satisfying

$$|F_\ell(\xi_1) - F_\ell(\xi_2)|_\ell \leq L_F \|\xi_1 - \xi_2\|_\ell \quad \text{for } \ell \in \mathbb{N},\ \xi_1, \xi_2 \in X_\ell.$$

(C8) For each $x, z \in X$ and each positive sequence $\{\nu_\ell\}$ convergent to 0 we have

$$\lim_{\ell\to\infty} |F_\ell(V_\ell x) - V_\ell F(x)|_\ell = 0,$$
$$\lim_{\ell\to\infty} |DF_\ell(V_\ell x) V_\ell z - V_\ell DF(x) z|_\ell = 0, \quad \text{and}$$
$$\lim_{\ell\to\infty} \Big(\sup_{\|\xi\|_\ell \leq \nu_\ell} |(DF_\ell(V_\ell x + \xi) - DF_\ell(V_\ell x)) V_\ell z|_\ell \Big) = 0.$$

Then we have

Theorem. *Let* (C1)-(C8) *be satisfied and* $F \in C^1(X, Y)$. *In addition we assume*

$$K(\alpha,\beta)L_F < 1 \quad \text{and} \quad \frac{M_2 M_3' K(\alpha,\beta) L_F}{1 - K(\alpha,\beta)L_F} < 1 \tag{G}$$

where

$$k(\alpha,\beta) = M\{M_1 M_2 \alpha^{-1} + M_4' \Gamma(1-\gamma)\beta^{\gamma-1} + M_5' \beta^{-1}\}, \tag{2.5}$$

$M_i' = M_i \max\{1, \underline{\lim}_{\ell\to\infty} \|W_\ell\|_{X_\ell, X}\}$ *for* $i = 3, 4, 5$ *and* Γ *denotes the gamma function. Then, (1.1) (resp. (1.2)) has an inertial manifold* $\mathcal{M}$ *(resp.* $\mathcal{M}_\ell$*) represented as a graph of a function* $h \in C^1(Y_1, P_2 X)$ *(resp.* $h_\ell \in C^1(Y_{\ell 1}, P_{\ell 2} X_\ell)$*). (See, e.g.,* [2], [5], [12] *for the definitions of the IMs.) Moreover, it holds that for every bounded set* $B \subset Y_1$

$$\lim_{\ell\to\infty} \sup_{y\in B} \|h_\ell(V_\ell y) - V_\ell h(y)\|_\ell = 0, \qquad \text{and} \tag{2.6}$$

$$\lim_{\ell\to\infty} \sup_{y\in B} \|Dh_\ell(V_\ell y) - V_\ell Dh(y)\|_{Y_1, X_\ell} = 0. \tag{2.7}$$

3. Proof

Existence: To prove the existence of IMs we use the results of [11] and [12] (also see [1]). By (C4) and (C5) the discrete version of the Trotter-Kato theorem (see [13, Theorem 6.7]) shows that $\bar{A}$, the closure of A in Y, generates a C_0-semigroup $\{S(t); t \ge 0\}$ on Y satisfying

$$\lim_{k_\ell \lambda_\ell \to t} |C(\lambda_\ell)^{k_\ell} V_\ell y - V_\ell S(t) y|_\ell = 0 \quad \text{for } y \in Y, t \ge 0. \tag{3.1}$$

In particular, (3.1) together with (C3) and (C4) implies that $|P_i S(t)y - S(t)P_i y| = \lim_{\ell\to\infty} |V_\ell(P_i S(t)y - S(t)P_i y)|_\ell = 0$, which shows that $P_i S(t) = S(t) P_i$ for $i = 1, 2$ and $t \ge 0$. Set $A_1 = \bar{A}|_{Y_1}$. Then, $D(A_1) = Y_1$ by (C5), so that $A_1 \in B(Y_1)$ by the closed graph theorem. The family $\{S_1(t)\}$ defined by $S_1(t) = S(t)|_{Y_1}$ forms a uniformly continuous group on Y_1 with the infinitesimal generator A_1. This proves conditions (S2) and (S3) in [11]. To show condition (S4) in [11] we fix $y \in Y$ and $t \ge 0$. By (2.2) and (3.1) we have one of the inequalities in (S4):

$$\begin{aligned} |S_1(-t)P_1 y| &= \lim_{k_\ell \lambda_\ell \to t} |V_\ell S_1(-t) P_1 y|_\ell \\ &\le M_2 \lim_{k_\ell \lambda_\ell \to t} e^{-(\alpha+\eta)k_\ell \lambda_\ell} |V_\ell y|_\ell = M_2 e^{-(\alpha+\eta)t} |y|. \end{aligned}$$

Moreover, we have by (2.4)

$$\begin{aligned} &\|W_\ell C(\lambda_\ell)^{k_\ell} P_{\ell 2} K_\ell V_\ell y\| \\ &\le \|W_\ell\|_{X_\ell, X} \{M_4((k_\ell+1)\lambda_\ell)^{-\gamma} + M_5\} e^{(\eta-\beta)k_\ell\lambda_\ell} |V_\ell y|_\ell, \end{aligned} \tag{3.2}$$

which implies that $\|W_\ell C(\lambda_\ell)^{k_\ell} P_{\ell 2} K_\ell V_\ell y\|$ is bounded as $k_\ell \lambda_\ell \to t > 0$. Note that by (3.1) $W_\ell C(\lambda_\ell)^{k_\ell} P_{\ell 2} K_\ell V_\ell y$ converges as $\ell \to \infty$ to $S(t) P_2 y$ in Y. Since X is reflexive and Y^* is dense in X^* by assumption, it also converges weakly in X to $S(t) P_2 y$. Then, passing to the limit in (3.2) yields

$$\|S(t) P_2 y\| \le \underline{\lim}_{\ell\to\infty} \|W_\ell\|_{X_\ell, X} \{M_4 t^{-\gamma} + M_5\} e^{(\eta-\beta)t} |y|,$$

the second inequality in (S4). Likewise, the remainder inequalities in (S4) will be proved.

Finally, let us prove condition (S1) in [11]. Note that $S(t)Y = P_1S(t)Y + S(t)P_2Y \subset X$ for $t > 0$ and $S(\cdot)x \in C(\mathbb{R}^+; Y)$ for $x \in X$. Since $\|S(t)x\| \le C\|x\|$ by (S4), we find that $S(\cdot)x$ is weakly continuous in X and hence $S(\cdot)x \in C(\mathbb{R}^+; X)$ (see [7, Theorem 10.6.5]). Thus (S1) is proved. Consequently, we can conclude from [11] and [1] that (1.1) has an IM, provided (G) is satisfied. The existence of an IM for (1.2) is already proved in [12] under the same assumptions as above.

Convergence: Next we prove (2.6) and (2.7). To this end recall ([12]) that the IM $\mathcal{M}$ for (1.1) is constructed as the graph of the function $h : Y_1 \to X$ defined by $h(y) = f(y, 0) - y$, where $f(y, t)$ is the unique function in $C^1(Y_1, C_{\eta+\varepsilon}(\mathbb{R}^+, X))$ for all $\varepsilon \in [0, \alpha)$ satisfying

(3.3)
$$f(y, s) = S(-t)y - \int_0^t S(s-t)P_1F(f(y, s))ds + \int_t^\infty S(s-t)P_2F(f(y, s))ds$$

for $y \in Y_1$ and $t \ge 0$. Here $C_\eta(\mathbb{R}^+, X)$ denotes the Banach space of continuous functions $u : \mathbb{R}^+ \to X$ with the norm $\|u\|^{(\eta)} = \sup_{t\ge 0} e^{\eta t}\|u(t)\|$. Similarly, the IM $\mathcal{M}_\ell$ for (1.2) is constructed as the graph of the function $h_\ell : Y_{\ell 1} \to X_\ell$ defined by $h_\ell(\xi) = \varphi_\ell(\xi, 0) - \xi$, where $\varphi_\ell(\xi, n)$ is the unique function in $C^1(Y_{\ell 1}, c_{\eta+\varepsilon}(\mathbb{N}, X_\ell))$ for all $\varepsilon \in [0, \alpha)$ satisfying

$$\begin{aligned}\varphi_\ell(\xi, n) = R_\ell^n\xi &- \lambda_\ell \sum_{i=1}^{n} R_\ell^{n-i+1}P_{\ell 1}K_\ell F_\ell(\varphi_\ell(\xi, i)) \\ &+ \lambda_\ell \sum_{i=n+1}^{\infty} Q_\ell^{i-n-1}P_{\ell 2}K_\ell F_\ell(\varphi_\ell(\xi, i))\end{aligned} \tag{3.4}$$

for $\xi \in Y_{\ell 1}$ and $n \ge 0$. Here $R_\ell = [C(\lambda_\ell)P_{\ell 1}]^{-1}$, $Q_\ell = C(\lambda_\ell)P_{\ell 2}$ and $c_\eta(\mathbb{N}, X_\ell)$ denotes the Banach space of bounded sequences $\tilde{x} = \{x_n\}_{n\ge 0}$ in X_ℓ with the norm $\|\tilde{x}\|_\ell^{(\eta)} = \sup_{n\ge 0} e^{\eta n\lambda_\ell}\|x_n\|_\ell$.

Since (2.6) is proved in [12], we shall show (2.7) only. Fix an arbitrary bounded set $B \subset Y_1$ and set for $y \in B$

$$\Omega_\ell(y) = \sup_{n\ge 0}\{\sup_{t\in G_{\ell n}} e^{\eta n\lambda_\ell}\|D\varphi_\ell(V_\ell y, n)V_\ell - V_\ell Df(y, t)\|_{Y_1, X_\ell}\} \tag{3.5}$$

where $G_{\ell n} = ((n-1)\lambda_\ell, n\lambda_\ell] \cap \mathbb{R}^+$. Here $Df(y, \cdot)$ denotes the Fréchet derivative of f at $y \in Y_1$, and so $Df(y, t) \in B(Y_1, X)$. Likewise, $D\varphi_\ell(\xi, n) \in B(Y_{\ell 1}, X_\ell)$ for $\xi \in Y_{\ell 1}$. To prove (2.7) it suffices to show

$$\lim_{\ell\to\infty} \sup_{y\in B} \Omega_\ell(y) = 0. \tag{3.6}$$

For $n \in \mathbb{N}, t \in G_{\ell n}$ and $y \in B$ we write

$$D\varphi_\ell(V_\ell y, n)V_\ell - V_\ell Df(y, t) = \sum_{i=1}^{8} H_i$$

with

$$H_1 = R_\ell^n V_\ell - V_\ell S(-n\lambda_\ell)P_1,$$
$$H_2 = -\sum_{i=1}^{n} \int_{G_{\ell i}} R_\ell^{n-i+1} K_\ell DF_\ell(\varphi_\ell(V_\ell y, i))\{D\varphi_\ell(V_\ell y, i)V_\ell - V_\ell Df(y,s)\}ds,$$
$$H_3 = -\sum_{i=1}^{n} \int_{G_{\ell i}} R_\ell^{n-i+1} K_\ell \{DF_\ell(\varphi_\ell(V_\ell y, i))V_\ell - V_\ell DF(f(y,s))\}Df(y,s)ds,$$
$$H_4 = -\sum_{i=1}^{n} \int_{G_{\ell i}} \{R_\ell^{n-i+1} K_\ell V_\ell - V_\ell S(s-n\lambda_\ell)P_1\}DF(f(y,s))Df(y,s)ds,$$
$$H_5 = \sum_{i=n+1}^{\infty} \int_{G_{\ell i}} Q_\ell^{i-n-1} P_{\ell 2} K_\ell DF_\ell(\varphi_\ell(V_\ell y, i))\{D\varphi_\ell(V_\ell y, i)V_\ell - V_\ell Df(y,s)\}ds,$$
$$H_6 = \sum_{i=n+1}^{\infty} \int_{G_{\ell i}} Q_\ell^{i-n-1} P_{\ell 2} K_\ell \{DF_\ell(\varphi_\ell(V_\ell y, i))V_\ell - V_\ell DF(f(y,s))\}Df(y,s)ds,$$
$$H_7 = \sum_{i=n+1}^{\infty} \int_{G_{\ell i}} \{Q_\ell^{i-n-1} P_{\ell 2} K_\ell V_\ell - V_\ell S(s-n\lambda_\ell)P_2\}DF(f(y,s))Df(y,s)ds,$$
$$H_8 = V_\ell(Df(y, n\lambda_\ell) - Df(y,t)).$$

By [12, Lemma 3.7] we have that for $\varepsilon > 0, \alpha + \varepsilon\eta > 0, z \in Y_1$ and $k, n \in \mathbb{N}$

$$e^{\eta n \lambda_\ell} \|H_1 z\|_\ell \le C_{\varepsilon,k}\rho_1(\ell, z) + C\rho_2(k, z)$$

with the functions $\rho_j(m, y), j = 1, 2$, such that the family $\{\rho_j(m, y)\}_{m\ge 1}$ is equicontinuous in $y \in Y$ and $\lim_{m\to\infty} \rho_j(m, y) = 0$ for each y. Here and in what follows, C denotes various constants and $C_{\varepsilon,k}$ denotes a constant depending on ε and k. Set $B_1 = \{z \in Y_1; |z| \le 1\}$. Since B_1 is compact in Y_1, we have $\lim_{m\to\infty} \sup_{z\in B_1} \rho_j(m, z) = 0$, and hence

$$\lim_{\ell\to\infty} \sup_{n\ge 0} e^{\eta n\lambda_\ell} \|H_1\|_{Y_1, X_\ell} = 0. \tag{3.7}$$

By a similar way as in [12, p.176] we can compute

$$e^{\eta n\lambda_\ell} \|H_2\|_{Y_1, X_\ell} \le e^{(\omega-\eta)\lambda_\ell} M M_1 M_2 L_F \alpha^{-1} \Omega_\ell(y). \tag{3.8}$$

For a fixed $T > 0$ take $T_\ell \in \mathbb{N}$ so that $T/\lambda_\ell - 1 < T_\ell \le T/\lambda_\ell$. Set

$$d_{\ell,T} = \sup |\{DF_\ell(\varphi_\ell(V_\ell y, i))V_\ell - V_\ell DF(f(y,s))\}Df(y,s)z|_\ell,$$

where the supremum is taken with respect to s, i, y and z satisfying $s \in G_{\ell 1}, 1 \le i \le T_\ell, y \in B$ and $z \in B_1$. Observing

$$\begin{aligned} &|\{DF_\ell(\varphi_\ell(V_\ell y, i))V_\ell - V_\ell DF(f(y,s))\}Df(y,s)z|_\ell \\ \le &|\{DF_\ell(\varphi_\ell(V_\ell y, i)) - DF_\ell(V_\ell f(y,s))\}V_\ell Df(y,s)z|_\ell \\ &+ |\{DF_\ell(V_\ell f(y,s))V_\ell - V_\ell DF(f(y,s))\}Df(y,s)z|_\ell \end{aligned}$$

and $\lim_{\ell\to\infty} \|\varphi_\ell(V_\ell y, i) - V_\ell f(y,s)\|_\ell = 0$ uniformly for $y \in B, s \in G_{\ell i}$ and $1 \le i \le T_\ell$ by (2.6) (also see [12, (4.3)]), we see that $\lim_{\ell\to\infty} d_{\ell,T} = 0$ by (C8) because the sets $\{f(y,s); y \in B, s \in [0,T]\}$ and $\{Df(y,s)z; y \in B, z \in B_1, s \in [0,T]\}$ are compact in X and Y, respectively. We also observe that $\|V_\ell\|_{X,X_\ell}$ and $\|V_\ell\|_{Y,Y_\ell}$ are bounded in ℓ by the uniform boundedness principle and that

$$\begin{aligned}\|Df(y,s)\|_{Y_1,X} \leq & e^{-(\eta+\varepsilon)i\lambda_\ell}\|Df(y)\|_{Y_1,C_{\eta+\varepsilon}(\mathbb{R}^+,X)} \\ \leq & Ce^{-(\eta+\varepsilon)i\lambda_\ell} \quad \text{for } s \in G_{\ell i} \text{ and } y \in B.\end{aligned} \tag{3.9}$$

Hence, by (C3)-(C7) $\|H_3 z\|_\ell$ for $z \in B_1$ is estimated by

$$Ce^{-(\alpha+\eta)n\lambda_\ell}\left\{d_{\ell,T}\sum_{i=1}^{T_\ell}\lambda_\ell e^{(\alpha+\eta)(i-1)\lambda_\ell} + L_F\sum_{i=T_\ell+1}^{n} e^{(\alpha+\eta)(i-1)\lambda_\ell}\int_{G_{\ell i}}\|Df(y,s)z\|ds\right\},$$

and so by (3.9)

$$e^{\eta n\lambda_\ell}\|H_3\|_{Y_1,X_\ell} \leq C(d_{\ell,T} + e^{-\varepsilon T}). \tag{3.10}$$

To estimate H_4 we take $z \in B_1$ and put $w = DF(f(y,s))Df(y,s)z$. By [12, Lemma 3.7] again $e^{\eta n\lambda_\ell}\|H_4 z\|_\ell$ is estimated by

$$\begin{aligned}& e^{\eta n\lambda_\ell}\sum_{i=1}^{T_\ell}\int_{G_{\ell i}} e^{-(n-i+1)(\alpha+\eta)\lambda_\ell/(1-\varepsilon)}(C_{\varepsilon,k}\rho_1(\ell,w) + C\rho_2(k,w))ds \\ & \qquad + Ce^{\eta n\lambda_\ell}\sum_{i=T_\ell+1}^{n}\int_{G_{\ell i}} e^{-(n\lambda_\ell+s)(\alpha+\eta)}L_F\|Df(y,s)z\|ds \\ & \leq e^{-(\alpha+\eta)T/(1-\varepsilon)}(C_{\varepsilon,k}\rho_1^*(\ell) + C\rho_2^*(k) + Ce^{-\varepsilon T}).\end{aligned}$$

Here we set

$$\rho_j^*(m) = \sup\{\rho_j(m,w); y \in B, z \in B_1, s \in [0,T]\},\ j = 1,2.$$

Since the set $\{w; y \in B, z \in B_1, s \in [0,T]\}$ is compact in Y, it holds that $\lim_{m\to\infty}\rho_j^*(m) = 0$. Hence, letting $\ell \to \infty, k \to \infty$ and $T \to \infty$ in this order, we get

$$\lim_{\ell\to\infty}\sup_{n\geq 1, y\in B} e^{\eta n\lambda_\ell}\|H_4\|_{Y_1,X_\ell} = 0. \tag{3.11}$$

By a similar manner as in [12, p.178] we can compute

$$e^{\eta n\lambda_\ell}\|H_5\|_{Y_1,X_\ell} \leq e^{(\beta-\eta)\lambda_\ell}L_F\{M_4\Gamma(1-\gamma)\beta^{\gamma-1} + M_5\beta^{-1}\}\Omega_\ell(y). \tag{3.12}$$

By using (2.4) one can estimate $\|H_6\|_{Y_1,X_\ell}$ by

$$\begin{aligned}& C\sum_{i=n+1}^{T_\ell}\lambda_\ell\{((i-n)\lambda_\ell)^{-\gamma} + 1\}e^{(\eta-\beta)(i-n-1)\lambda_\ell}d_{\ell,T} \\ & \quad + C\sum_{i=T_\ell+1}^{\infty}\int_{G_{\ell i}}\{((i-n)\lambda_\ell)^{-\gamma} + 1\}e^{(\eta-\beta)(i-n-1)\lambda_\ell}L_F\|Df(y,s)\|_{Y_1,X}ds.\end{aligned}$$

Hence, by (3.9)

$$e^{\eta n\lambda_\ell}\|H_6\|_{Y_1,X_\ell} \leq C(d_{\ell,T} + e^{-\varepsilon T}). \tag{3.13}$$

Next, by using [12, Lemma 3.10] we obtain for $z \in B_1$ and $v = DF(f(y,s))Df(y,s)z$

$$\|H_7 z\|_\ell \le \int_0^{T-n\lambda_\ell} (s^{-\gamma}+1)e^{(1-\varepsilon)(\eta-\beta)s}(C_{\varepsilon,k}\sigma_1(\ell,v) + C\sigma_2(k,v))ds$$
$$+ C\int_{T-n\lambda_\ell}^{\infty} (s^{-\gamma}+1)e^{(\eta-\beta)s}\|Df(y,s)z\|ds$$

with the functions $\sigma_j(m.w)$, $j=1,2$, such that the family $\{\sigma_j(m,w)\}_{m\ge 1}$ is equicontinuous in $w \in Y$ and $\lim_{m\to\infty}\sigma_j(m,w) = 0$ for each w. Hence, by (3.9)

$$e^{\eta n\lambda_\ell}\|H_7\|_{Y_1,X_\ell} \le C_{\varepsilon,k}\sigma_1^*(\ell) + C\sigma_2^*(k) + Ce^{-\varepsilon T}. \tag{3.14}$$

Here we set

$$\sigma_j^*(m) = \sup\{\sigma_j(m,v); y\in B, z\in B_1, s\in[0,T]\},\ j=1,2.$$

Just as in the case of ρ_j^* we see that $\lim_{m\to\infty}\sigma_j^*(m) = 0$.

Finally, set

$$\delta_T(h) = \sup\|Df(y,s)z - Df(y,\hat{s})z\|$$

where the supremum is taken over all $y \in B$, $z \in B_1$ and $s, \hat{s} \in [0,2T]$ with $|s-\hat{s}| \le h$. It is easy to see that $\lim_{h\downarrow 0}\delta_T(h) = 0$. A similar computation as in [12, p.179] yields that for $z \in B_1, y \in B$

$$e^{\eta n\lambda_\ell}\|H_8 z\|_\ell \le C(\delta_T(\lambda_\ell) + e^{-\varepsilon T}\|Df(y)\|_{Y_1,C_{\eta+\varepsilon}(\mathbb{R}^+,X)}). \tag{3.15}$$

We are now in a position to prove (3.6). By virtue of (3.7), (3.8) and (3.10)-(3.15) we obtain

$$\overline{\lim}_{\ell\to\infty}\sup_{y\in B}\Omega_\ell(y) \le K(\alpha,\beta)L_F\,\overline{\lim}_{\ell\to\infty}\sup_{y\in B}\Omega_\ell(y).$$

Since $K(\alpha,\beta)L_F < 1$ by (G), we conclude that (3.6) holds.

4. Example

We briefly consider the renormalized Kuramoto-Sivashinsky equation (KSE)

$$u_t + D^4u + D^2u + uDu = 0, \qquad (x,t)\in\mathbb{R}\times\mathbb{R}^+,$$

subject to periodic boundary condition, with period L. We refer to [4], [6], [12] in the notation and some results concerning the KSE. We view it as an evolution equation in the Hilbert space $Y = \{u \in L^2_{per}(0,L); u \text{ is odd}\}$ with the usual L^2 norm. Since the KSE in Y has a bounded absorbing set in $X = H^2_{per}(0,L)\cap Y$, i.e., there exists a constant $r_0 > 0$ such that for every $r > 0$ we can choose a time $T^*(r) > 0$ satisfying $\|u(t)\|_{H^2} \le r_0$ for all $t \ge T^*(r)$ and all $u(0)\in Y$ with $\|u(0)\|_{L^2} \le r$, we may consider the prepared equation instead of the KSE

$$du/dt = Au + Fu, \qquad t\in\mathbb{R}^+, \tag{4.1}$$

where $Au = -D^4u$ and $Fu = -D^2u - \rho(\|u\|_{H^2})uDu$ with a smooth function ρ satisfying $0\le\rho\le 1$ and $\rho(r) = 1$ if $|r|\le r_0$, $\rho(r) = 0$ if $|r|\ge 2r_0$.

Let Y_ℓ be the Hilbert space $S^\ell_{odd,per}$, the set of vectors $\xi = (\xi_0,\cdots,\xi_{\ell-1})$ satisfying $\xi_0 = 0$ and $\xi_j = \xi_{\ell-j}$ for $j = 1,\cdots,\ell-1$, with norm $|\xi|_\ell = (h\sum_{k=1}^{\ell-1}\xi_k^2)^{1/2}, h = L/\ell$. For convenience ξ will be extended periodically to an infinite sequence by $\xi_{j+\ell} = \xi_j$. Define $\Delta_\ell : Y_\ell \to Y_\ell$ by

$$(\Delta_\ell\xi)_k = h^{-2}(\xi_{k-1} - 2\xi_k + \xi_{k+1}) \qquad \text{for} \quad \xi\in Y_\ell,$$

where $(\Delta_\ell \xi)_k$ denotes the k-th element of $\Delta_\ell \xi$. We also consider $S^\ell_{odd,per}$ as a Hilbert space with norm $\|\xi\|_\ell = |\Delta_\ell \xi|_\ell$, which is denoted by X_ℓ.

We approximate (4.1) by the finite difference scheme

$$\lambda_\ell^{-1}(\xi^{i+1} - \xi^i) + \Delta_\ell^2((1-\theta)\xi^i + \theta\xi^{i+1}) - F_\ell(\xi^i) = 0, \quad \xi^i \in Y_\ell, \quad i \in \mathbb{N} \tag{4.2}$$

$(1/2 < \theta \le 1)$, where $F_\ell(\xi) = -\Delta_\ell \xi - \rho(\|\xi\|_\ell^2)B^\ell(\xi)$ and $B^\ell : Y_\ell \times Y_\ell \to Y_\ell$ is defined by

$$(B^\ell(\xi))_k = (6h)^{-1}(\xi_{k-1} + \xi_k + \xi_{k+1})(\xi_{k+1} - \xi_{k-1}) \qquad \text{for } \xi \in Y_\ell.$$

Then, (4.2) is rewitten as the form (1.2) by setting

$$C(\lambda_\ell) = (I - (1-\theta)\lambda_\ell \Delta_\ell^2)(I + \theta\lambda_\ell\Delta_\ell^2)^{-1} \text{ and } K_\ell = (I + \theta\lambda_\ell\Delta_\ell^2)^{-1}.$$

By some operational calculi and spectral theorems one finds that conditions (C1)-(C8) hold with $M = M_2 = M_3 = 1, M_4 = 2, \gamma = 1/2, \omega = 0, M_1 = \nu_N^{1/2}, M_5 = (2\nu_{N+1})^{1/2}, \alpha = \beta = (\nu_{N+1} - \nu_N)/4, \eta = -(\nu_{N+1} + \nu_N)/2$ for $N \in \mathbb{N}$. Here we set $\nu_k = (2\pi k/L)^4$, the eigenvalues of the operator A in Y. Hence, if N is large, then condition (G) is satisfied and consequently we can apply our theorem.

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