PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 4, April 1999, Pages 1209–1213
S 0002-9939(99)05216-8

THE MOD 2 HOMOLOGY OF BSO

CLAUDINA IZEPE RODRIGUES

(Communicated by Ralph Cohen)

Abstract. This note is about a set of generators to the mod 2 homology of BSO.

1. Introduction

It is well known that $H_*(BO;\mathbb{Z}_2)$ is a polynomial ring, $\mathbb{Z}_2[x_i|i \geq 1]$, where $x_i \in H_i(BO;\mathbb{Z}_2)$. The generators x_i may be chosen to come from the nonzero classes in $H_i(BO_1;\mathbb{Z}_2)$ under the stabilization map, and in particular, $x_i = f_*[RP^i]$, where $f : RP^i \to BO$ classifies the usual line bundle over projective space.

The corresponding dual basis of $H^*(BO;\mathbb{Z}_2)$ is usually denoted by s_w, where $w = (i_1, ..., i_r)$. If the splitting principle is used to write universal Stiefel-Whitney classes w_i formally as the i-th elementary symmetric function in 1-dimensional classes $t_1, t_2, ...,$ then $s_w = \sum t_1^{i_1} t_2^{i_2} ... t_r^{i_r}$ is the smallest symmetric function containing the given monomial. In particular, if $y = \sum a_{j_1 \ldots j_s} x_1^{j_1} ... x_s^{j_s} \in H_*(BO;\mathbb{Z}_2)$, the coefficients are $a_{j_1 \ldots j_s} = s_{(j_1, \ldots, j_s)}[y]$.

S. Papastavridis [1] has shown that $H_*(BSO;\mathbb{Z}_2)$ is also a polynomial ring, $\mathbb{Z}_2[y_i|i > 1]$, which is described as a subring of $H_*(BO;\mathbb{Z}_2)$ by choosing classes y_i as polynomials in the x_j. (Note: It is well known that $H^*(BSO;\mathbb{Z}_2)$ is the quotient of $H^*(BO;\mathbb{Z}_2)$ by the ideal generated by w_1. Dually, $H_*(BSO;\mathbb{Z}_2)$ can be identified with a subring of $H_*(BO;\mathbb{Z}_2)$.) Papastavridis' choices of the classes y_i are clearly algebraically independent and hence give a subring of $H_*(BO;\mathbb{Z}_2)$ which has precisely the same dimension as $H_*(BSO;\mathbb{Z}_2)$. The hard part of his argument is to see that the classes y_i lie in $H_*(BSO;\mathbb{Z}_2)$.

The purpose of this paper is to simplify Papastavridis' argument. For any integer $n > 1$, one chooses a pair of integers (j,k) with $j + k = n$ by

$$\begin{cases} (j,k) = (0,n) & \text{if} \quad n = 2^r, \\ (j,k) = (2^r, 2^{r+1}s) & \text{if} \quad n = 2^r(2s+1),\ s > 0. \end{cases}$$

Then, let $z_n \in H_n(BO;\mathbb{Z}_2)$ be the classes $f_*[RP^j \times RP^k]$ where $f : RP^j \times RP^k \to BO$ classifies the bundle $\xi_1 \oplus \xi_2 \oplus (\xi_1 \otimes \xi_2)$ with ξ_i being the usual line bundle over the i-th factor. Because the given bundle is orientable, it is clear that $z_n \in \text{image}(H_n(BSO;\mathbb{Z}_2) \to H_n(BO;\mathbb{Z}_2))$, and our main result is

Theorem. $H_*(BSO;\mathbb{Z}_2) = \mathbb{Z}_2[z_n|n > 1]$.

Received by the editors June 17, 1997.
1991 *Mathematics Subject Classification.* Primary 55R40.

©1999 American Mathematical Society

Additionally, one has:

Fact. The classes $z_n = f_*[RP^j \times RP^k]$ coincide with Papastavridis' classes y_n.

I wish to express my thanks to Professor R. E. Stong for suggesting this problem.

2. Proof of the Theorem

It is clear that one has a homomorphism

$$\varphi : \mathbb{Z}_2[u_n | n > 1] \to H_*(BSO;\mathbb{Z}_2) \subset H_*(BO;\mathbb{Z}_2)$$

defined by $\varphi(u_n) = z_n = f_*[RP^j \times RP^k]$, and in every dimension, $H_*(BSO;\mathbb{Z}_2)$ and the polynomial ring have the same dimension as the $\mathbb{Z}_2$ vector space. To prove the theorem, it suffices to see that the classes z_n are algebraically independent. This is immediate from:

Lemma.

$$z_n = \begin{cases} x_n + \textit{decomposables} & \textit{if} \quad n = 2^r(2s+1), \\ \\ x_{n/2}^2 & \textit{if} \quad n = 2^r. \end{cases}$$

Proof. Let $H^*(RP^j \times RP^k;\mathbb{Z}_2) = \mathbb{Z}_2[\alpha,\ \beta]/(\alpha^{j+1} = 0, \beta^{k+1} = 0)$, where $dim\ \alpha = dim\ \beta = 1$. The Stiefel-Whitney class of $\xi_1 \oplus \xi_2 \oplus (\xi_1 \otimes \xi_2)$ is $(1+\alpha)(1+\beta)(1+\alpha+\beta)$. Then for $n = 2^r(2s+1)$,

$$\begin{aligned} s_n &= \alpha^n + \beta^n + (\alpha+\beta)^n \\ &= \binom{2^r(2s+1)}{2^r} \alpha^{2^r} \beta^{2^{r+1}s} \ , \end{aligned}$$

which is nonzero. For $n = 2^r$, $\alpha = 0$, and $H^*(RP^n;\mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{n+1} = 0)$, with the Stiefel-Whitney class of the bundle being $(1+\beta)^2$. Then

$$s_w((1+\beta)^2) = \begin{cases} 0 & \text{if} \quad w \neq (w', w'), \\ \\ s_{w'}((1+\beta)^2) & \text{if} \quad w = (w', w'), \end{cases}$$

giving $z_n = x_{n/2}^2$.

3. Papastavridis' classes

To verify that $z_n = y_n$, as defined by Papastavridis, requires a lot of unpleasant calculation. Not only is one showing that y_n belongs to $H_n(BSO;\mathbb{Z}_2)$, but one is identifying the given class. Since this is obvious for $n = 2^r$, one need only consider $n = 2^r(2s+1)$. The goal is to verify that $s_{(a,b,c)}[z_n]$, with $0 \le a \le b \le c$, is given by Papastavridis' formula,

$$\begin{cases} \binom{b-1}{2^r - a - 1} & \text{if} \quad 2^r \le b \ \text{ and } \ 0 \le a < 2^r, \\ \\ \binom{b-1}{r(2^r - a)} & \text{if} \quad 0 < b < 2^r,\ 0 \le a < 2^r \ \text{ and } \ a + r(2^r - a) \le b, \\ \\ 0 & \text{otherwise.} \end{cases}$$

It is, of course, clear that $s_w[z_n] = 0$ if $w = (i_1, ..., i_r)$ with $r > 3$, since the defining bundle has dimension 3.

Here we are going to verify that $s_{(a,b,c)}[z_n]$ is given by the above formula only in case $0 < a < b < 2^r$, $c < 2^{r+1}s$. In this case we have

$$\begin{aligned} s_{(a,b,c)}[z_n] &= \{\alpha^a\beta^b(\alpha+\beta)^c + \alpha^a\beta^c(\alpha+\beta)^b + \alpha^b\beta^c(\alpha+\beta)^a \\ &\quad +\alpha^b\beta^a(\alpha+\beta)^c\}[RP^{2^r} \times RP^{2^{r+1}s}] \ , \ \text{ since } \ \alpha^c = 0, \\ &= \left[\binom{c}{2^r-a} + \binom{b}{2^r-a} + \binom{a}{2^r-b} + \binom{c}{2^r-b}\right] \quad \bmod \ 2. \end{aligned}$$

Next, observe that

$$\binom{a}{2^r-b}$$

is the coefficient of x^{2^r-b} in the binomial expansion of $(1+x)^a$, and $(1+x)^a = \dfrac{(1+x)^{2^{r+1}s}}{(1+x)^{b+c-2^r}}$, with $b+c-2^r-1 > 0$. Recall that $\dfrac{1}{(1+y)^{t+1}} = \displaystyle\sum_{j=0}^{\infty}\binom{t+j}{j}y^j$. Then

$$\binom{a}{2^r-b} \equiv \binom{b+c-2^r-1+2^r-b}{2^r-b} \equiv \binom{c-1}{2^r-b} \qquad \bmod \ 2 \ .$$

So, it is clear that

$$\begin{aligned} \binom{c}{2^r-b} + \binom{a}{2^r-b} &\equiv \binom{c}{2^r-b} + \binom{c-1}{2^r-b} \\ &\equiv \binom{c-1}{2^r-b-1} = \binom{2^{r+1}s+2^r-a-b-1}{2^{r+1}s-a}, \end{aligned}$$

and this is the coefficient of $x^{2^{r+1}s-a}$ in the binomial expansion of $(1+x)^{c-1}$, where

$$\begin{aligned} (1+x)^{c-1} &= \frac{(1+x)^{2^{r+1}s}(1+x)^{2^r}}{(1+x)^{a+b+1}} \\ &= \Bigg\{ \underbrace{\sum_{m=0}^{s}\binom{s}{m}x^{2^{r+1}s-2^{r+1}m}}_{\text{(A)}} + \underbrace{\sum_{m=0}^{s}\binom{s}{m}x^{2^{r+1}s-2^{r+1}m+2^r}}_{\text{(B)}} \Bigg\} \\ &\quad \times \Bigg\{ \underbrace{\sum_{l=0}^{\infty}\binom{a+b+l}{l}x^l}_{\text{(C)}} \Bigg\}. \end{aligned}$$

Now, if we take the product of any term in (A) by the complementary term in (C), the coefficient is

$$\binom{s}{m}\binom{2^{r+1}m+b}{2^{r+1}m-a} \equiv 0 \quad \bmod \ 2,$$

since $b < 2^r$. (Obs: if $m = 0$ in (A), $2^{r+1}s > 2^{r+1} - a$.) Next, observe that all the powers in the expansion (B) for $m > 0$ are between $2^{r+1}(s-1)+2^r$ and $2^{r+1}s$, moreover, we have $2^{r+1}(s-1)+2^r < 2^{r+1}s - a < 2^{r+1}s$, since $2^r > a > 0$. For

$m = 0$, the power is $2^{r+1}s + 2^r$, so it is bigger than $2^{r+1}s$. Therefore,

$$\binom{c}{2^r - b} + \binom{a}{2^r - b} = \left\{ \sum_{m=1}^{s} \binom{s}{m} \binom{2^{r+1}m - 2^r + b}{2^{r+1}m - 2^r - a} \right\}$$
$$= \left\{ \sum_{m=1}^{s} \binom{s}{m} \right\} \binom{b}{2^r - a} \equiv \binom{b}{2^r - a} \mod 2.$$

Thus, we can see immediately that

$$s_{(a,b,c)}[z_n] = \binom{c}{2^r - a} \mod 2.$$

Now, writing $2^r - a = 2^j + t$, $0 \leq t < 2^j$, with $r(2^r - a) = t$ as in Papastavridis [1], we have $c = 2^{r+1}s + 2^j + t - b$. So, we can look at $\binom{c}{2^r - a}$ as the coefficient of $x^{2^r - a}$ in the binomial expansion of

$$(1+x)^c = \frac{(1+x)^{2^{r+1}s}(1+x)^{2^j}}{(1+x)^{b-t}}.$$

Hence, it follows that

$$s_{(a,b,c)}[z_n] = \left\{ \binom{2^j + b - 1}{2^j + t} + \binom{b-1}{t} \right\} \mod 2.$$

Next, we can write $b = a + b'$ with $0 < b' < 2^r - a = 2^j + t$, since $a < b < 2^r$. Thus, we get

$$\binom{2^j + b - 1}{2^j + t} = \binom{2^j + 2^j(2^{r-j} - 1) - t + b' - 1}{2^j + t}$$
$$\equiv \begin{cases} \binom{b-1}{t} \mod 2 & \text{if } 0 < b' \leq t, \\ 0 \mod 2 & \text{if } t < b' < 2^j + t \end{cases}$$

and

$$\binom{b-1}{t} = \binom{2^j(2^{r-j} - 1) - t + b' - 1}{t} \equiv 0 \mod 2 \quad \text{if } t < b' < 2t.$$

Finally, since

$$0 < b' < 2t \iff a + 2r(2^r - a) > b$$

and

$$2t \leq b' < 2^j + t \iff a + 2r(2^r - a) \leq b,$$

we conclude that

$$s_{(a,b,c)}[z_n] = \begin{cases} 0 \mod 2 & \text{if } a + 2r(2^r - a) > b, \\ \binom{b-1}{r(2^r - a)} \mod 2 & \text{if } a + 2r(2^r - a) \leq b \end{cases}$$

as in [1].

Since we have checked one of the more difficult cases, one now can write down the other possibilities for (a, b, c) using similar calculations.

References

[1] S. Papastavridis, The image of $H_*(BSO;\mathbb{Z}_2)$ in $H_*(BO;\mathbb{Z}_2)$, Proc. Amer. Math. Soc., Vol. 107, No. 4, 1989, pp. 1071-1073. MR **90m:**55017

IMECC - Departamento de Matemática, Universidade Estadual de Campinas, Caixa Postal - 6065, 13081-970 - Campinas - S.P., Brasil

E-mail address: `claudina@ime.unicamp.br`