PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1339–1345
S 0002-9939(99)04621-3
Article electronically published on January 28, 1999

ASYMPTOTICS OF THE D'ALEMBERTIAN WITH POTENTIAL ON A PSEUDO-RIEMANNIAN MANIFOLD

THOMAS BRANSON AND GESTUR ÓLAFSSON

(Communicated by Palle E. T. Jorgensen)

Abstract. Let $\Box$ be the Laplace-d'Alembert operator on a pseudo-Riemannian manifold (M, g). We derive a series expansion for the fundamental solution $G(x, y)$ of $\Box + H$, $H \in C^\infty(M)$, which behaves well under various symmetric space dualities. The qualitative properties of this expansion were used in our paper in *Invent. Math.* **129** (1997), 63–74, to show that the property of vanishing logarithmic term for $G(x, y)$ is preserved under these dualities.

1. Introduction

Let (M, g) be a pseudo-Riemannian manifold of dimension n, with Laplace-d'Alembert operator $\Box$. Let H be a smooth function on M, and consider the problem of constructing a fundamental solution for the d'Alembertian with potential $\Box + H$.

A classical construction of Hadamard formally develops the fundamental solution of $\Box + H$ in the form

$$G(x, y, \Box + H) = \begin{cases} U(x, y)\sigma^{(2-n)/2} + V(x, y)\log|\sigma|, & n \text{ even}, \\ U(x, y)|\sigma|^{(2-n)/2}, & n \text{ odd}, \end{cases} \tag{1}$$

for (x, y) in a neighborhood $\mathcal{O}$ of the diagonal in $M \times M$, where $\sigma = \sigma(x, y)$ is the geodesic distance-squared, and U and V are smooth functions on $\mathcal{O}$. Of course, the distance-squared may be negative, since the metric may be indefinite. The true fundamental solution is a version of the formal expression (1) which is regularized, either in a classical sense [H], or in a distributional sense [C], [F].

The precise analytic considerations needed to produce the fundamental solution vary according to the metric signature. This subject of this paper is a classical development of the series (1) which, for certain purposes, is a valuable alternative to the Hadamard development. In particular, we used the existence of this development in an essential way in [BO]. There we proved that the vanishing of the logarithmic term V is preserved under various symmetric space dualities. This, in turn, allowed us to construct many new locally symmetric spaces on which V vanishes, taking the potential H to be a constant multiple of the scalar curvature. The property of vanishing logarithmic term has been studied by many authors; for metrics of Lorentz

Received by the editors July 8, 1997 and, in revised form, August 6, 1997.
1991 *Mathematics Subject Classification.* Primary 47F05.
Research of both authors partially supported by NSF grants.
Research of the second author partially supported by a LEQSF grant.

signature, it is equivalent to *Huygens' principle*; for Riemannian signature, there are interpretations with consequences for classical gravitation and electrostatics. In this paper, we limit ourselves to signature-independent considerations and develop the required expansion algebraically, without treating convergence questions. In particular, all computations take place off $\{\sigma = 0\}$.

Note that in this generality, the construction also produces formal asymptotic expansions of the resolvent kernels of the same operators $\Box + H$; i.e., of the kernel functions for the $(\Box + H - \lambda)^{-1}$, where $\lambda \in \mathbb{C}$.

In normal coordinates with origin at y, let (x^α) be the coordinates of the moving point x. The coefficients V_k of the Taylor series for V,

$$\begin{aligned} V(x,y) \sim V_0(y) + (V_1)_\alpha(y)x^\alpha + (V_2)_{\alpha\beta}(y)x^\alpha x^\beta + \dots \\ + (V_k)_{\alpha_1\dots\alpha_k}(y)x^{\alpha_1}\dots x^{\alpha_k} + \dots , \end{aligned}$$

are universal local invariants of the metric g and the potential H, valued in the symmetric k-tensor fields on M. (Here and below, the summation convention is in force.) Similar considerations hold for the function U. In fact, the Taylor coefficients for U and V may be calculated inductively and algebraically from the Taylor expansions of the metric g and the potential H.

Choose normal coordinates for g in which $y = 0$ and $x = (x^\alpha)$. It will be convenient to introduce an artificial flat reference metric: Let η be the standard flat metric of the same signature as g. Then g has a normal coordinate expansion

$$\begin{aligned} g_{\alpha\beta}(x) &= \eta_{\alpha\beta} + g_{\alpha\beta,\gamma\delta}(y)x^\gamma x^\delta + \dots + g_{\alpha\beta,\gamma_1\dots\gamma_k}(y)x^{\gamma_1}\dots x^{\gamma_k} + O(|x|^{k+1}), \\ g^{\alpha\beta}(x) &= \eta^{\alpha\beta} + g^{\alpha\beta}{}_{,\gamma\delta}(y)x^\gamma x^\delta + \dots + g^{\alpha\beta}{}_{,\gamma_1\dots\gamma_k}(y)x^{\gamma_1}\dots x^{\gamma_k} + O(|x|^{k+1}), \end{aligned} \tag{2}$$

where the $|\cdot|$ in $O(|x|^{k+1})$ refers to any positive definite metric. The tensors $g_{\alpha\beta,\gamma_1\dots\gamma_k}$ and $g^{\alpha\beta}{}_{,\gamma_1\dots\gamma_k}$ are local invariants; that is, they are linear combinations of monomials

$$\mathcal{C}(g \otimes \dots \otimes g \otimes g^\sharp \otimes \dots \otimes g^\sharp \otimes (\nabla \dots \nabla R) \otimes \dots \otimes (\nabla \dots \nabla R)), \tag{3}$$

where $g^\sharp = (g^{\alpha\beta})$ is the inverse of the metric $g = (g_{\alpha\beta})$, ∇ is the metric connection, R is the Riemann curvature tensor of ∇, and $\mathcal{C}$ is a contraction operator. As a consequence of the metric expansion (2), the normal coordinate metric determinant $\mathbf{g}$ has a similar expansion:

$$\mathbf{g}(x) = \pm 1 + \mathbf{g}_{,\alpha\beta}(y)x^\alpha x^\beta + \dots + \mathbf{g}_{,\alpha_1\dots\alpha_k}(y)x^{\alpha_1}\dots x^{\alpha_k} + O(|x|^{k+1}),$$

where the tensors $\mathbf{g}_{,\alpha_1\dots\alpha_k}$ are linear combinations of monomials of the form (3).

By Weyl's invariant theory [W], the local invariants V_k are linear combinations of monomials of the form

$$\begin{aligned} \mathcal{C}(g \otimes \dots \otimes g \otimes g^\sharp \otimes \dots \otimes g^\sharp \otimes (\nabla \dots \nabla R) \otimes \dots \otimes (\nabla \dots \nabla R) \\ \otimes (\nabla \dots \nabla H) \otimes \dots \otimes (\nabla \dots \nabla H)). \end{aligned} \tag{4}$$

By taking note of the behavior of all terms under uniform dilation $g' = sg$, $0 < s \in \mathbb{R}$, it is easy to compute that each monomial (4) in V_k (resp. U_j) enjoys a homogeneity property:

$$p_\nabla + 2(p_R + p_H) = n - 2 + k \qquad (\text{resp. } p_\nabla + 2(p_R + p_H) = j),$$

where p_∇ (resp. p_R, p_H) is the number of ∇ (resp. R, H) appearing. Note that implementation of the Ricci identities may convert occurrences of ∇ into occurrences of R, but does not disturb the quantity $p_\nabla + 2(p_R + p_H)$.

2. The expansion

Thus far, it has not been necessary to fix sign conventions on $\Box$ and R; we do so now. $\Box_M f$ is the total contraction of $-g^\sharp \otimes \nabla\nabla f$. In a local frame (X_α),

$$R^\alpha{}_{\beta\gamma\delta} X_\alpha = R(X_\gamma\,, X_\delta)X_\beta\,.$$

In local coordinates, the d'Alembertian is

$$\Box = -|\mathbf{g}|^{-1/2}\partial_\beta(g^{\alpha\beta}|\mathbf{g}|^{1/2}\partial_\alpha) =: -\eta^{\alpha\beta}\partial_\alpha\partial_\beta + q^{\alpha\beta}(x)\partial_\alpha\partial_\beta + b^\alpha(x)\partial_\alpha\,. \tag{5}$$

In normal coordinates, the vanishing of the first order terms in the metric expansions gives

$$q^{\alpha\beta}(x) = O(|x|^2), \qquad b^\alpha(x) = O(|x|). \tag{6}$$

Let $D = -\eta^{\alpha\beta}\partial_\alpha\partial_\beta$. Our strategy will be to explicitly compute the effect of D on homogeneous terms of the expansion (1), and to qualitatively observe the effect of $\Box + H - D$. We break up a homogeneous term according to

$$(V_k)_{\alpha_1\ldots\alpha_k} x^{\alpha_1}\ldots x^{\alpha_k} = V_k(x^{(k)}) = \sum_{\ell=0}^{[k/2]} \sigma^\ell V_{k,\ell}(x^{(k-2\ell)}), \tag{7}$$

where $x^{(k)}$ is the k-tuple (of n-tuples) $(x, \ldots, x)$, and the $V_{k,\ell}$ are trace free tensors. (That is, any contraction of $V_{k,\ell}$ with $\eta^\sharp$ vanishes.) In representation theoretic terms, we can accomplish this by taking the symmetric tensor representation of $\mathrm{GL}(n,\mathbb{R})$ and decomposing into irreducible representations of the subgroup $\mathrm{O}(p,q)$, where (p,q) is the signature of η. We similarly decompose each homogeneous term in the expansion of U.

For simplicity, we restrict to $\{\sigma > 0\}$ for purposes of this calculation. After insertion of appropriate minus signs, it is clear that the situation on $\{\sigma < 0\}$ is also described.

D obeys the second order Leibniz rule $D(\varphi\psi) = \varphi D\psi + \psi D\varphi - 2\eta^\sharp(d\varphi, d\psi)$. But if Ω is an arbitrary trace free symmetric p tensor and $\psi = \Omega(x^{(p)})$,

$$D\psi = 0, \qquad \eta^\sharp(d\sigma, d\psi) = 2p\psi. \tag{8}$$

Furthermore, $D\sigma = -2n$ and $\eta^\sharp(d\sigma, d\sigma) = 4\sigma$, so that for $s \in \mathbb{C}$,

$$\begin{aligned} D(\sigma^s) &= -2s(n+2s-2)\sigma^{s-1}, \\ D(\sigma^s \log\sigma) &= -2\sigma^{s-1}\{s(n+2s-2)(\log\sigma) + n + 4s - 2\}. \end{aligned} \tag{9}$$

Putting all this together, we get (for ψ as above)

$$\begin{aligned} D(\sigma^s\psi) &= -2s(n+2s+2p-2)\sigma^{s-1}\psi, \\ D(\sigma^s(\log\sigma)\psi) &= -2s(n+2s+2p-2)\sigma^{s-1}(\log\sigma)\psi - 2(n+4s+2p-2)\sigma^{s-1}\psi. \end{aligned}$$

Thus for any $t \in \mathbb{C}$, we have

$$\begin{aligned} &D(\sigma^{\ell-t}U_{j,\ell}(x^{(j-2\ell)})) = -2(\ell-t)(n-2\ell-2t+2j-2)\sigma^{\ell-t-1}U_{j,\ell}(x^{(j-2\ell)}), \\ &D(\sigma^\ell(\log\sigma)V_{k,\ell}(x^{(k-2\ell)})) \\ &\qquad = \{-2\ell(n-2\ell+2k-2)\sigma^{\ell-1}\log\sigma - 2(n+2k-2)\sigma^{\ell-1}\}V_{k,\ell}(x^{(k-2\ell)}). \end{aligned}$$

In particular, setting $t = m := (n-2)/2$, we have

$$D(\sigma^{\ell-m} U_{j,\ell}(x^{(j-2\ell)})) = -4(\ell-m)(j-\ell)\sigma^{\ell-m-1} U_{j,\ell}(x^{(j-2\ell)}),$$
$$D(\sigma^{\ell}(\log\sigma) V_{k,\ell}(x^{(k-2\ell)})) = -4\sigma^{\ell-1}\{\ell(k-\ell+m)(\log\sigma)+k+m\} V_{k,\ell}(x^{(k-2\ell)}).$$

The important qualitative point about $\Box + H - D$ is (6). We have:

Lemma 1. *On $\{\sigma > 0\}$, for formal power series U and V as in (7),*

$$\begin{aligned}(\Box + H)(\sigma^{-m}U + (\log\sigma)V) &= 2n(n-2)\sigma^{-m-1}U_0 \\ &+ \sigma^{-m-1}\sum_{j=1}^{\infty}\left\{ W_j - \sum_{\ell=0}^{[j/2]} 4(\ell-m)(j-\ell)\sigma^{\ell} U_{j,\ell}(x^{(j-2\ell)})\right\} \\ &- \sigma^{-1}\sum_{k=0}^{\infty}\sum_{\ell=0}^{[k/2]} 4(k+m)\sigma^{\ell} V_{k,\ell}(x^{(k-2\ell)}) \\ &+ \sigma^{-1}(\log\sigma)\sum_{k=0}^{\infty}\left\{\tilde{W}_k - \sum_{\ell=0}^{[k/2]} 4\ell(k-\ell+m)\sigma^{\ell} V_{k,\ell}(x^{(k-2\ell)})\right\}\end{aligned}$$

where W_j (resp. $\tilde{W}_k$) is a homogeneous polynomial of degree j (resp. k). $W_{j,\ell}$ depends on $\{U_u\}$ and $\{V_v\}$ only through $\{U_{u,L} \mid u < j,\ L \le \ell\}$ and $\{V_{v,L} \mid v < j-m,\ L \le \ell\}$. $\tilde{W}_{k,\ell}$ depends on $\{U_u\}$ and $\{V_v\}$ only through $\{V_{v,L} \mid v < k,\ L \le \ell\}$.

Theorem 2. *If n is odd, given a constant U_0, the formula of Lemma 1, with $V = 0$, inductively computes a unique formal power series solution to the equation $(\Box + H)(U|\sigma|^{-m}) = 0$ on $\{\sigma \ne 0\}$. The coefficients $U_{j,l}(y)$ are universal local invariants as in (4). If n is even, the formula of Lemma 1(b) inductively computes a formal power series solution to $(\Box + H)(U\sigma^{-m} + V\log|\sigma|) = 0$ on $\{\sigma \ne 0\}$. This solution is unique modulo solutions without singularity at $\sigma = 0$. With the side conditions $U_{j,m} = 0$ for all j, the even-dimensional solution is unique, and all coefficients $U_{j,l}(y)$ and $V_{k,l}(y)$ are universal local invariants as in (4).*

Proof. First restrict to $\{\sigma > 0\}$. Proceeding inductively, we find by examining the logarithm free terms that the $U_{j,\ell}$ are uniquely determined for n odd. If n is even,

1. The $U_{j,\ell}$ are uniquely determined for $\ell < m$ and all j;
2. The $U_{j,m}$ may be prescribed arbitrarily, but the $V_{k,0}$ are uniquely determined for all k.

Switching attention to the terms with a $\log\sigma$ factor, we get no additional condition on the $V_{k,0}$, and

3. The $V_{k,\ell}$ are uniquely determined for $\ell > 0$ and all k.

Going back to the logarithm free terms,

4. The $U_{j,\ell}$ for $\ell > m$ are uniquely determined, given our prescription of the $U_{j,m}$.

If E and F are two power series constructed as above, differing in the prescription of the $U_{j,m}$ (and its effects on the computation of the $U_{j,\ell}$ for $\ell > m$), then $E - F$ is a power series (without singularity at $\sigma = 0$) satisfying $(\Box + H)(E - F) = 0$. Thus the power series construction is unique up to the addition of such *harmonics*.

It is straightforward to insert signs as appropriate to extend the conclusion to $\{\sigma < 0\}$. The local invariance properties follow inductively from those of the metric, through (5). $\square$

The first term on the right in Lemma 1(a) regularizes to a constant multiple of the delta function when we take account of behavior through $\{\sigma = 0\}$, and compute in the sense of distributions. Thus we are computing the fundamental solution of $\Box + H$ by implementing the above procedure. Since $\Box + H - \lambda$ is an operator of the same type, our results also cover the *resolvent kernel*, i.e., the kernel function of $(\Box + H - \lambda)^{-1}$.

3. Comparison with the Hadamard expansion

Some remarks on the relation of the above results to the Hadamard expansion are in order. In Hadamard's original treatment, $U(x, y)$ and $V(x, y)$ are expanded in series of two point functions:

$$U(x,y) \sim \sum_{j=0}^{\infty} K_j(x,y)\sigma^j, \qquad V(x,y) \sim \sum_{k=0}^{\infty} L_k(x,y)\sigma^k.$$

The K_j and L_k are determined, in a neighborhood of the diagonal in $M \times M$, by recursive solution of the *transport equations*; these are ordinary differential equations in which the geodesic parameter is the independent variable.

With x as the moving point and y the fixed point, the definition of $\Box$ gives the following variants of (9). Let $\mathbf{G} := \log|\mathbf{g}|$ and $r = \sigma^{1/2}$, and let a prime denote d/dr. Then

$$\begin{aligned}\Box\sigma^s &= -4s(m + s + \tfrac{1}{4}r\mathbf{G}')\sigma^{s-1},\\ \Box(\sigma^s \log\sigma) &= -2\sigma^{s-1}\{s(n + 2s - 2 + \tfrac{1}{2}r\mathbf{G}')(\log\sigma) + n + 4s - 2 + \tfrac{1}{2}r\mathbf{G}'\}.\end{aligned}$$

This gives

$$\begin{aligned}(\Box + H)(K_j\sigma^{j-m}) &= ((\Box + H)K_j)\sigma^{j-m}\\ &\quad - 4(j-m)\sigma^{j-m-1}(rK_j' + \{j + \tfrac{1}{4}r\mathbf{G}'\}K_j),\\ (\Box + H)(L_k\sigma^k \log\sigma) &= ((\Box + H)L_k)\sigma^k \log\sigma\\ &\quad - 2k\sigma^{k-1}\log\sigma(2rL_k' + \{n + 2k - 2 + \tfrac{1}{2}r\mathbf{G}'\}L_k)\\ &\quad - 2\sigma^{k-1}(2rL_k' + \{n + 4k - 2 + \tfrac{1}{2}r\mathbf{G}'\}L_k).\end{aligned}$$

The analysis of, e.g., [C], [F] shows that, after suitable regularization and normalization of U_0,

$$(\Box + H)(\sigma^{-m}U + (\log\sigma)V) = \delta_y(x) + (\text{smooth}).$$

We need to make sure that $(\Box + H)(\sigma^{-m}U + (\log\sigma)V)$ vanishes to infinite order for $x \neq y$. The bottom (σ^{-m-1}) coefficient gives the condition

$$rK_0' + \tfrac{1}{4}r\mathbf{G}'K_0 = 0;$$

this shows that up to normalization,

$$K_0(x,y) = \left(\frac{\mathbf{g}(y)}{\mathbf{g}(x)}\right)^{1/4}.$$

Proceeding inductively, we then get the conditions

$$4(j-m)(rK_j' + \{j + \tfrac{1}{4}r\mathbf{G}'\}K_j) = (\Box + H)K_{j-1}\,, \qquad 0 < j < m\,.$$

When we reach the σ^{-1} coefficient in $(\Box + H)(\sigma^{-m}U + (\log\sigma)V)$, we get no condition on K_m, but rather

$$2(2rL_0' + \{n - 2 + \tfrac{1}{2}\mathbf{G}'\}L_0) = (\Box + H)K_{m-1}\,.$$

Treating the coefficients in the $\log\sigma$ series inductively, we get

$$2k(2rL_k' + \{n + 2k - 2 + \tfrac{1}{2}\mathbf{G}'\}L_k) = (\Box + H)L_{k-1}\,, \qquad k \geq 1.$$

Treating the L_k as known and prescribing K_m arbitrarily, we then "clean up" by solving the equations

$$\begin{aligned} &4k(rK_{k+m}' + \{k + m + \tfrac{1}{4}r\mathbf{G}'\}K_{k+m}) \\ &\qquad = (\Box + H)K_{k+m-1} - 2(2rL_k' + \{n + 4k - 2 + \tfrac{1}{2}r\mathbf{G}'\}L_k). \end{aligned}$$

Through this, the ambiguity in K_m propagates to the expansion of an arbitrary harmonic summand; this is of course the expected non-uniqueness.

Several ordinary differential equations of the form $ru' + bu = f$, $b(r) = b_0 + O(r^2)$, appeared in the above discussion. If $b_0 > 0$, solutions of the homogeneous equation $ru' + bu = 0$ are singular at $r = 0$, so a nonsingular solution of $ru' + bu = f$, if any, will be unique. Setting $u = r^{-b_0}y$, we get

$$y' + \beta y = r^{b_0 - 1}f, \qquad \beta := r^{-1}(b - b_0) = O(r).$$

To avoid a singularity in u at $r = 0$, we must take $y(0) = 0$, so that

$$y(r) = \exp\left(-\int_0^r \beta(s)ds\right)\int_0^r \exp\left(\int_0^s \beta(z)dz\right)s^{b_0-1}f(s)ds.$$

If f is nonsingular at $r = 0$ and b_0 is a positive integer, then $y(r) = O(r^{b_0})$. Thus u is nonsingular at $r = 0$. Since n is even, inspection of the process which produces the functions K_j and L_k shows that they are uniquely determined and nonsingular at $r = 0$.

When one actually tries to calculate the local invariants in the Hadamard expansion, attention quickly turns to the Taylor expansions of the Hadamard coefficients. The K_j and L_k do not have trace free Taylor coefficients; thus these coefficients do not just come from the list of $U_{j,\ell}$ and $V_{k,\ell}$ produced by our power series construction. However, the Taylor coefficients $L_{k,p}$ of the L_k can be computed from the list $V_{k,\ell}$ and *vice versa*; similarly for the lists $K_{j,p}$ and $U_{j,\ell}$. The computation of any given entry from one list involves only finitely many entries from the counterpart list. The separation of V_k into the various $V_{k,\ell}$ can be carried out effectively using (8,9); that is, by taking the eigenresolution of σD on the space of k homogeneous polynomials. The ambiguity in the definition of the $U_{j,m}$ (resp. $K_{m,p}$) affects only the $U_{j,q}$ (resp. the $K_{q,p}$) for $q \geq m$, and has no effect on the logarithmic terms.

References

[BO] T. Branson and G. Ólafsson, *Helmholtz operators and symmetric space duality*, Invent. Math. **129** (1997), 63–74. CMP 97:16

[C] E. Combet, *Solutions Élémentaires des Dalembertians Généralisées*, Mém. Sc. Math. Facs. CLX, Gauthier-Villars, Paris, 1965.

[F] F. Friedlander, *The Wave Equation on a Curved Space-time*, Cambridge University Press, Cambridge, 1975. MR **57:**889

[H] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover Publications, New York, 1952. MR **14:**474f

[W] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, Princeton, 1939.

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242
E-mail address: `branson@math.uiowa.edu`

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803
E-mail address: `olafsson@marais.math.lsu.edu`