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ON h-COBORDISMS OF SPHERICAL SPACE FORMS

SŁAWOMIR KWASIK AND REINHARD SCHULTZ

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Abstract. Given a manifold M of dimension at least 4 whose universal covering is homeomorphic to a sphere, the main result states that a compact manifold W is isomorphic to a cylinder $M \times [0,1]$ if and only if W is homotopy equivalent to this cylinder and the boundary is isomorphic to two copies of M; this holds in the smooth, PL and topological categories. The result yields a classification of smooth, finite group actions on homotopy spheres (in dimensions ≥ 5) with exactly two singular points.

In the topology of manifolds it is often important to recognize when a manifold W with boundary is isomorphic to a cylinder $M \times [0,1]$. The s-cobordism theorem gives the standard principle for recognizing products, but in some situations it is useful to have other criteria involving the boundary components of W. In a series of papers [U1]–[U3] F. Ushitaki has studied this question for equivariant h-cobordisms between two free linear G-spheres $S(V)$, $S(V')$ of dimension $2n-1 \geq 5$ and has proved that such h-cobordisms are equivariantly isomorphic to products $S(V) \times I$ under assumptions of an algebraic nature (*e.g.*, the vanishing of $SK_1(\mathbf{Z}[G])$). To be more specific, the following is the main result.

Theorem (Ushitaki)**.** *Let G be a finite group, and X a free G-homotopy sphere of dimension $2n-1 \geq 5$. Then the following are equivalent.*

(1) *Every smooth G-h-cobordism W between X and itself must be G-diffeomorphic to $X \times I$.*

(2) *The homomorphism*

$$\tilde{c} : H_{2n}\left(\mathbf{Z}_2; Wh(G)^{\text{trivial}}\right) \to L^s_{2n}(G)$$

in the Rothenberg exact sequence is a monomorphism.

Here one takes the standard conjugation involution on the Whitehead group $Wh(G)$ corresponding to the trivial homomorphism $G \to \mathbf{Z}_2$.

Let G be a finite group which can act freely (topologically, piecewise linearly or smoothly) on a homotopy sphere Σ^n. We shall call the manifold $\Sigma^n/G = M^n$ a *fake spherical space form* (*cf.* [KS4]). In this note we elaborate on the techniques and results of our paper [KS3] and obtain the following:

Theorem. *Let CAT denote one of the topological, piecewise linear, or smooth categories, let $G \neq \{1\}$ be a finite group, and let M^n be a CAT manifold that is a fake*

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spherical spaceform Σ^n/G, where $n \geq 4$. Then every CAT h-cobordism W with $\partial W \approx M^n \sqcup M^n$ is CAT-isomorphic to $M^n \times I$.

We shall say that an h-cobordism W is *inertial* if ∂W is a disjoint union of two copies of the same manifold.

Corollary 1. *Every equivariant CAT h-cobordism between free linear G-spheres $S(V)$ and $S(V')$ of dimension n is equivariantly CAT isomorphic to $S(V) \times I$ if $n \geq 4$.*

The results of [CS1]–[CS2] and [KS2], [KS5] show that the theorem and corollary do not extend to topological h-cobordisms when $n = 3$.

Corollary 1 settles the cases left open in [U1]–[U3], and it also yields an answer to a question studied by M. Sebastiani [Se, Theorems on pp. 437–438] on the classification and enumeration of smooth semifree actions of a finite group G on a homotopy sphere Σ^n (where $n \geq 6$) with exactly two fixed points.

Corollary 2. *Every smooth, semifree action of a finite group G on a homotopy sphere Σ^n, $n \geq 5$, with exactly two fixed points is smoothly equivalent to a twisted double of the form $D(V) \cup_f D(V)$. Here $D(V)$ is a disk in a free representation V of G and f is an equivariant diffeomorphism of the boundary sphere $S(V)$. In particular the number of equivariant diffeomorphism classes of such actions is finite.*

Remarks. (1) This result was proved in [Se] for $G \approx \mathbf{Z}_k$, where k is odd or $k = 2$.

(2) Corollary 2 does not extend to smooth semifree G-actions on homotopy spheres with 1-dimensional fixed point sets; more precisely, such smooth G-manifolds need not have presentations as twisted doubles of the form $D(W) \cup_h D(W)$ for some linear representation W. The construction of examples will be outlined at the end of this paper.

(3) Partial analogs of Corollary 2 for continuous actions are discussed in [KS1].

One way of proving the main result if $n \geq 5$ (and also if $n = 4$ in the topological category) would be to show the triviality of $\ker \tilde{c}$ directly and then to use Ushitaki's result. Instead, we shall give an essentially self-contained account that shortens some of the arguments in [U1]–[U3] and [KS2]–[KS3].

Proof of the Theorem. The usual case is when $n \geq 5$ in the smooth or PL category, or $n \geq 4$ in the topological category. In these cases the relevant h-cobordisms are classified by elements of the Whitehead group. The only cases not included are those where $n = 4$ in the smooth or PL category (the two cases are equivalent because every PL manifold of dimension ≤ 6 has a unique equivalence class of smoothings). We shall first prove the result in the usual cases, and afterwards we shall give the proof in the exceptional cases.

The usual cases. Let G be a finite group which acts freely on Σ^n, where the action is topological, piecewise linear or smooth as appropriate. The action of G is orientation preserving unless n is even and $G \approx \mathbf{Z}_2$, and in the latter cases all h-cobordisms are products because $Wh(\mathbf{Z}_2) = 0$. Therefore we shall assume for the rest of the discussion of the usual case that G acts orientation preservingly and n is odd. Since G has periodic homology, the classification of such groups in [TW] and [DM] shows that every 2-hyperelementary subgroup H of G is either

(a) a metacyclic group given by a semi-direct product $\mathbf{Z}_n \times_T \mathbf{Z}_{2^e}$, or

(b) a semi-direct product $\mathbf{Z}_n \times_T Q(2^k)$, where $k \geq 3$ [DM, Remark 5.5, p. 273].

When $H \approx \mathbf{Z}_n \times_T \mathbf{Z}_{2^e}$, then the group $SK_1(\mathbf{Z}[H]) = 0$ [O1, p.190], but for $H \approx \mathbf{Z}_n \times_T Q(2^k)$, $SK_1(\mathbf{Z}[H])$ is nontrivial [O1, p.198]. If G is a finite periodic group, then $SK_1(\mathbf{Z}[G])$ is an elementary abelian 2-group: *i.e.*, $SK_1(\mathbf{Z}[G]) \approx (\mathbf{Z}_2)^\ell$ for some $\ell \geq 0$ (*cf.* [O2], Theorem 6, p. 334). Consequently Theorem 2.1 in [KS3] can be strengthened as follows.

Claim 1. *If G is a finite periodic group, then the standard conjugation involution on $Wh(G)$ associated to the trivial homomorphism $G \to \mathbf{Z}_2$ is the identity.*

The point here is that in analogy with the proof of Theorem 2.1 in [KS3] it is enough to show the triviality of the involution only when G is 2-hyperelementary.

If G is metacyclic, then $Wh(G)$ is torsion free and the involution is trivial by the result of C.T.C. Wall [W2, Theorem 6.1]. When G is given by $\mathbf{Z}_n \times_T Q(2^k)$ then the triviality of the involution $* : Wh(G) \longrightarrow Wh(G)$ is proved in [KS3, Thm. 2.1].

Now let $(W^{n+1}; M^n, M^n)$ be a topological ($n \geq 4$) or smooth ($n \geq 5$) inertial h-cobordism with M^n a fake spherical space form. Let $\tau = \tau(W^{n+1}, \partial_0 W^{n+1} = M^n)$ be the Whitehead torsion of this h-cobordism. It was already shown in [KS3] that τ must be an element in $SK_1(\mathbf{Z}[G]) = \mathrm{Torsion}\, Wh(G)$.

Claim 2. *There is a homotopy equivalence of triads*

$$f : (W^{n+1}; M^n, M^n) \longrightarrow (M^n \times I; M^n \times \{0\}, M^n \times \{1\})$$

such that (1) $\tau(f) = -\tau$, (2) $f|_{\partial W^{n+1}} : (M^n \sqcup M^n) \longrightarrow (M^n \times \{0\} \sqcup M^n \times \{1\})$ *is the identity.*

Proof of Claim 2. Property (1) is already established in [KS3, Cor. 3.3]. With respect to property (2) it will suffice to verify that, in our situation, each homotopy self-equivalence of M^n is homotopic to the identity if $\pi_1(M^n) \approx G \not\approx \mathbf{Z}_2$. First of all we can assume that $G \not\approx \mathbf{Z}_2$, for if $G \approx \mathbf{Z}_2$, then each h-cobordism is an s-cobordism (because $Wh(\mathbf{Z}_2) = 0$) and the conclusion follows by the s-cobordism theorem. Given then that $G \not\approx \mathbf{Z}_2$, the equivariant Hopf theorem [tD, Thm. 8.4.1, p. 213] implies that indeed each self-homotopy equivalence is homotopic to the identity.

Consider the following commutative diagram in surgery theory:

(∗)

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & L^s_{2k+1}(G) & \xrightarrow{\gamma^s} & S^s_{\mathrm{CAT}}(M^n \times I, \partial) & \xrightarrow{\eta^s} & [\Sigma M^n; F/\overline{\mathrm{CAT}}] & \xrightarrow{\theta^s} & L^s_{2k}(G) \\
& \big\downarrow \ell_1 & & t \big\downarrow & & \big\| id & & \big\downarrow \ell_0 \\
\ldots \longrightarrow & L^h_{2k+1}(G) & \xrightarrow{\gamma^h} & S^h_{\mathrm{CAT}}(M^n \times I, \partial) & \xrightarrow{\eta^h} & [\Sigma M^n; F/\overline{\mathrm{CAT}}] & \xrightarrow{\theta^h} & L^h_{2k}(G)
\end{array}$$

Here $\overline{\mathrm{CAT}}$ denotes one of TOP , PL, or O, depending upon whether CAT stands for the TOP, PL, or DIFF category. The vertical maps in this diagram come from the Rothenberg exact sequence (*cf.* [W1]).

Claim 3. *The homomorphism $\ell_1 : L^s_{2k+1}(G) \longrightarrow L^h_{2k+1}(G)$ is onto.*

Proof of Claim 3. The Dress induction theorem [Dr, Thm. 1, p. 293] implies

(a) $L^s_{2k+1}(G) = \varprojlim_{H \subset G} L^s_{2k+1}(H)$,

(b) $L^h_{2k+1}(G) = \varprojlim_{H \subset G} L^h_{2k+1}(H)$

where H runs over the conjugacy classes in G of 2-hyperelementary subgroups, and the maps are the restrictions. It follows easily that one then has to show the surjection $L^s_{2k+1}(H) \longrightarrow L^h_{2k+1}(H)$ only when H is 2-hyperelementary (there are only finitely many groups involved in these inverse limits). As is observed in the proof of Lemma 3.4 in [KS3] it is enough to consider the case of $H \approx \mathbf{Z}_n \times_T Q(2^k)$ (and in fact the only nontrivial considerations occur for $H \approx \mathbf{Z}_n \times Q(2^k)$). However this case is handled by [KS3]. Strictly speaking, [KS3, Lemma 3.4] deals only with k even (*i.e.*, $L^{s,h}_{4k+1}$-groups) but everything remains valid if k is odd.

Given the surjectivity of the homomorphism $\ell_1 : L^s_{2k+1}(G) \longrightarrow L^h_{2k+1}(G)$ one proceeds as follows. Let $[f] \in S^h_{\mathrm{CAT}}(M^n \times I, \partial)$ be the element determined by the map of triads $f : (W^{n+1}; M^n, M^n) \longrightarrow (M^n \times I; M^n \times \{0\}, M^n \times \{1\})$. If $[f] \in \mathrm{Image}\,\gamma^h$, then the surjectivity of $\ell_1 : L^s_{2k+1}(G) \longrightarrow L^h_{2k+1}(G)$ implies that f is h-cobordant (relative boundary) to f', where $[f'] \in S^s_{\mathrm{CAT}}(M^n \times I, \partial)$. To be more specific, write $f' : (\overline{W}^{n+1}; M^n, M^n) \longrightarrow (M^n \times I; M^n, M^n)$, where $\overline{W}^{n+1}$ is an s-cobordism; there is an h-cobordism X^{n+2} with $\partial X^{n+2} = \overline{W}^{n+1} \cup -W^{n+1}$ and $\partial \overline{W}^{n+1} \cap -\partial W^{n+1} = M^n_0 \sqcup M^n_1$ (where $M^n_0, M^n_1 \approx M^n$). By the duality formula for Whitehead torsions of h-cobordisms we have

$$\tau(X^{n+2}, \overline{W}^{n+1}) = \tau(X^{n+2}, W^{n+1})^*.$$

Since $\tau(X^{n+2}, W^{n+1})^* = \tau(X^{n+2}, W^{n+1})$, by Claim 1 it follows that $\tau(X^{n+2}, \overline{W}^{n+1}) = \tau(X^{n+2}, W^{n+1})$. Furthermore, we also have

$$\tau(W^{n+1}, M^n_0) = \tau(X^{n+2}, \overline{W}^{n+1}) + \tau(\overline{W}^{n1}, M_0) - \tau(X^{n+2}, W^{n+1}).$$

Since $\overline{W}^{n+1}$ is an s-cobordism, the middle term vanishes so that $\tau(W^{n+1}, M^n_0) = \tau(X^{n+2}, \overline{W}^{n+1}) - \tau(X^{n+2}, W^{n+1}) = \tau(X^{n+2}, W^{n+1}) - \tau(X^{n+2}, W^{n+1}) = 0$.

Suppose now $[f] \notin \mathrm{im}\,\gamma^h$ so that $|[f]| = \eta^h[f] \in [\Sigma M^n; G/\overline{\mathrm{CAT}}]$ is nontrivial. Because $f|_{\partial W^{n+1}}$ is the identity, it follows that

$$\theta^s(|[f]|) \in L^s_{2k}(G_2) \subset L^s_{2k}(G),$$

where G_2 is the 2-Sylow subgroup of G (*cf.* [W3]).

Claim 4. $\theta^s(|[f]|) = 0$.

Proof of Claim 4. From the diagram $(*)$ we infer that $\ell_0 \theta^s(|[f]|) = 0$; in particular $\theta^s|[f]| \in \ker \ell_0 \in H^{2k+1}(\mathbf{Z}_2; Wh(G))$ and in fact the element $\theta^s(|[f]|)$ can be identified with the element $-\tau = \tau(f) \in SK_1(\mathbf{Z}[G])$. On the other hand a 2-Sylow subgroup G_2 of G is either a cyclic group $\mathbf{Z}_{2^\ell}$ or a quaternionic group $Q(2^k)$, and in both cases $SK_1(\mathbf{Z}[G_2]) = 0$ (*cf.* [O1]). Consequently $\theta^s(|[f]|) = 0$ and hence there exists an element $[\widetilde{f}] \in S^s_{\mathrm{CAT}}(M^n \times I, \partial)$ such that $\eta^s[\widetilde{f}] = |[f]|$.

Conclusion of the proof in the usual cases. Given Claim 4, the proof of our theorem in the usual cases can be completed by observing that $t[\widetilde{f}] = [f]$. Namely, this once again means that f is h-cobordant (relative boundary) to a simple homotopy equivalence $f' \in S^s_{\mathrm{CAT}}(M^n \times I, \partial)$ which leads to the vanishing of the Whitehead torsion $\tau = \tau(W^{n+1}; M^n)$.

Proof of the theorem in the exceptional cases. A result of D. Barden [Ba] states that every smooth inertial h-cobordism with boundary $S^4 \sqcup S^4$ is a product (see [Sh, Thm. 6.1, pp. 348-349] for a proof); more generally, if W^5 is a smooth, oriented, simply connected inertial h-cobordism such that the oriented boundary of W^5 has the form $M^4 \sqcup -M^4$ for some oriented smooth homotopy sphere M^4, then the same argument shows that W^5 is also a product. We cannot quite say whether every inertial h-cobordism in this case is a product because of the unresolved status of the smooth 4-dimensional Poincaré Conjecture (there might be a smooth fake 4-sphere that is not orientation reversingly diffeomorphic to itself). On the other hand, if $n = 4$, then the only possible fundamental group of a nonsimply connected fake spherical spaceform is $\mathbf{Z}_2$, and thus in the exceptional case it suffices to show that every inertial smooth h-cobordism $(W^5; M^4, M^4)$ with M^4 homotopy equivalent to $\mathbf{RP}^4$ is a product.

The argument is very similar to that of [Sh, §6] and [KS2, §2]. Let $i_0, i_1 : M^4 \to W^5$ be the inclusions of the boundary components, and let $r : W^5 \to M^4$ be a homotopy inverse to i_0. By the existence of collar neighborhoods for boundaries we may as well assume that the composite $r \circ i_0$ is the identity. On the other hand, it is well known that every homotopy self-equivalence of $\mathbf{RP}^4$ is homotopic to the identity, and therefore it follows that the homotopy self-equivalence $r \circ i_1$ is also homotopic to the identity; we may as well assume as before that $r \circ i_1$ is the identity. If

$$\mu : (W^5; M^4, M^4) \to ([0,1]; \{0\}, \{1\})$$

is a Morse function, then the map

$$(r, \mu) : (W^5; M^4, M^4) \to (M^4 \times [0,1]; M^4 \times \{0\}, M^4 \times \{1\})$$

defines a homotopy equivalence that is a diffeomorphism on both ends; *i.e.*, an element of the relative structure set $S_{\mathrm{DIFF}}(M^4 \times I, \partial)$; the s superscript is omitted because $Wh(\mathbf{Z}_2) = 0$. Since this structure set involves smooth surgery problems where the boundary is untouched, one can analyze this structure set using the usual methods of surgery theory. Therefore consider the corresponding surgery exact sequence of groups:

$$[\Sigma^2(M^4_+); F/O] \xrightarrow{\sigma} L^s_6(\mathbf{Z}_2^-) \xrightarrow{\Delta} S_{\mathrm{DIFF}}(M^4 \times I, \partial) \xrightarrow{\mathbf{N}} [\Sigma(M^4_+); F/O] \longrightarrow L^s_5(\mathbf{Z}_2^-).$$

The groups $L^s_5(\mathbf{Z}_2^-)$ and $L^s_6(\mathbf{Z}_2^-)$ are isomorphic to 0 and $\mathbf{Z}_2$ respectively. Furthermore, the map σ is onto; in particular, if the relative surgery problem $h : (V^2, S^1) \to (D^2, S^1)$ has Kervaire invariant 1, then the product formulas for surgery obstructions imply that

$$h \times \mathrm{id}(M^4) : (V \times M^4, S^1 \times M^4) \to (D^2 \times M^4, S^1 \times M^4)$$

represents a class $u \in [\Sigma^2(M^4_+), F/O]$ such that $\sigma(u) \neq 0$. Therefore the map $\mathbf{N}$ is bijective. Direct computation shows that $[\Sigma(M^4_+), F/O] \approx H^2(\Sigma(M^4_+); \mathbf{Z}_2) \approx \mathbf{Z}_2$, the isomorphism being given by the Sullivan class $k_2 : F/O \to K(\mathbf{Z}_2, 2)$.

Let $\mathcal{E}(M^4)$ be the topological monoid of homotopy self-equivalences of M^4 based at the identity. As in [KS2, §2] there is a canonical homomorphism

$$J_M : \pi_1(\mathcal{E}(M^4)) \to S^s_{\mathrm{DIFF}}(M^4 \times I, \partial) \to [\Sigma(M^4_+), F/O]$$

and the same considerations as in the proof of [KS2, Thm. 2.1, Case II, pp. 531–532] show that W^5 must be a smooth product if J_M is onto.

Suppose first that $M^4 \approx \mathbf{RP}^4$. In this case one can prove surjectivity as in [KS2, Thm. 2.1, pp. 530–532]. More precisely, if $\xi : S^5 \to S^4$ denotes the Hopf map, the argument shows that the composite

$$\nabla(\xi) : \mathbf{RP}^4 \times I \xrightarrow{\text{pinch}} \mathbf{RP}^4 \times I \vee S^5 \xrightarrow{\text{id}\vee\xi} \mathbf{RP}^4 \times I \vee S^4$$
$$\xrightarrow{\text{id}\vee\text{proj}} \mathbf{RP}^4 \times I \vee \mathbf{RP}^4 \xrightarrow{\text{fold}} \mathbf{RP}^4 \times I$$

defines a nontrivial normal invariant in $[\Sigma(\mathbf{RP}^4), F/O]$. This completes the proof if M^4 is (diffeomorphic to) $\mathbf{RP}^4$.

To prove the general case, we first note that $\pi_1(\mathcal{E}(M^4))$ and $\pi_1(\mathcal{E}(\mathbf{RP}^4))$ are canonically isomorphic. Specifically, if $a : M^4 \to \mathbf{RP}^4$ is a homotopy equivalence and b is a homotopy inverse to a, then the correspondence is given by sending $\psi : I \times \mathbf{RP}^4 \to \mathbf{RP}^4$ into the composite

$$\psi_M := b \circ \psi \circ (\text{id} \times a).$$

Let Ψ and Ψ_M be the associated homotopy self-equivalences of $I \times \mathbf{RP}^4$ and $I \times M^4$ given by (proj_I, ψ) and (proj_I, ψ_M) respectively; by construction the restrictions of both maps to the boundary are the identity, and therefore both determine relative homotopy smoothings. We claim that the normal invariants satisfy $\mathbf{N}(\Psi_M) = (\text{id} \times a)^*\mathbf{N}(\Psi)$. If so, then the nontriviality of $\mathbf{N}(\Psi)$ for suitably chosen ψ (as in the special case $M^4 \approx \mathbf{RP}^4$) will imply the nontriviality of $\mathbf{N}(\Psi_M)$, and since $[\Sigma(M^4_+), F/O] \approx \mathbf{Z}_2$ it will follow that J_M is onto. As noted earlier, this will prove the theorem in the exceptional case for M^4.

To determine the relationship between $\mathbf{N}(\Psi_M)$ and $(\text{id} \times a)^*\mathbf{N}(\Psi)$, consider a more general situation. Suppose we are given a homotopy self-equivalence f of the manifold with boundary V such that the restriction to ∂V is the identity, and suppose we are also given the homotopy equivalence $A : (U, \partial U) \to (V, \partial V)$ such that the map of boundaries is a diffeomorphism. Let B be a homotopy inverse to A such that the induced map of boundaries is a diffeomorphism. Then the standard formula for the normal invariant of a composite (*cf.* [Sh]) implies that

$$\mathbf{N}(B \circ f \circ A) = A^*\mathbf{N}(f) + \left(A^* \circ ((f^*)^{-1} - 1)\right)\mathbf{N}(A) \in [U/\partial U, F/O].$$

In the case of interest to us here, this means that $\mathbf{N}(\Psi_M) = (\text{id} \times a)^*\mathbf{N}(\Psi)$ if Ψ induces the identity on $[\Sigma(\mathbf{RP}^4_+), F/O]$. But the latter is isomorphic to $\mathbf{Z}_2$, so the automorphism Ψ^* must be the identity. This completes the proof of the theorem in the exceptional cases.

Proof of Corollary 1. Let $S(V), S(V')$ be free linear G-spheres which are equivariantly h-cobordant. Results of M. Atiyah and R. Bott (*cf.* [Mi, p. 409]) show that $S(V)$ is equivariantly isometric with $S(V')$. Consequently the h-cobordism is inertial and Corollary 1 follows from the main theorem.

Proof of Corollary 2. The splitting of Σ^n as a twisted double $D(W) \cup_f D(W)$ follows directly from the main theorem. Since the equivariant diffeomorphism type of the twisted double is uniquely determined by the pseudo-isotopy class of the equivariant diffeomorphism $f : S(W) \longrightarrow S(W)$ and the latter is uniquely determined by the pseudo-isotopy class of $f/G : S(W)/G \longrightarrow S(W)/G$, it suffices to know that the set of all such pseudo-isotopy classes is finite. As noted in the proof of Claim 2, the map f is equivariantly homotopic to the identity if the order of G is ≥ 3, and if $G \approx \mathbf{Z}_2$, then f is equivariantly homotopic to either the identity or a hyperplane reflection. Thus in all cases there are finitely many homotopy classes of self-diffeomorphisms of $S(W)/G$ that induce the identity on the fundamental group, and it suffices to show that there are only finitely many pseudo-isotopy classes of self-diffeomorphisms that are homotopic to the identity. The group of all such classes is a quotient of the structure set $S^s_{\mathrm{DIFF}}(S(V)/G \times I, \partial)$, and inspection of the surgery exact sequence shows that the latter is finite; therefore the number of pseudo-isotopy classes is finite as required.

Remarks. (1) The assumption that the h-cobordism in our theorem be inertial is clearly necessary, for the realization of Whitehead torsions by h-cobordisms leads to nontrivial h-cobordisms between distinct fake spherical spaceforms. Various questions concerning inertial h-cobordisms and their Whitehead torsions were studied in [H].

(2) One can also use the diagram $(*)$ in the proof of the main theorem to shorten some arguments in [KS2]–[KS3].

Examples. To see that Corollary 2 does not extend to smooth semifree G-actions on homotopy spheres with 1-dimensional fixed point sets, suppose that G has order ≥ 3, let W be a semifree G-representation of odd dimension ≥ 7 with 1-dimensional fixed point set, and let $\alpha \in L^h_{\dim W+1}(G)$ be arbitrary. By [BQ] there is a transverse linear, G-isovariant normal cobordism X from the linear G-sphere $S(W \oplus R)$ to another G-manifold Y such that Y is transverse linearly isovariantly homotopy equivalent to $S(W \oplus R)$ and the relative surgery obstruction for the map of triads $X \to S(W \oplus R) \times I$ is α. A diagram chase as in [Sc, §6] shows that the equivariant Whitehead torsion of the map $Y \to S(W \oplus R)$ represents the image of α under the homomorphism $\tau : L^h_{\dim W+1}(G) \to H_*(\mathbf{Z}_2; Wh(G))$ in the Rothenberg exact sequence relating L^s to L^h. On the other hand, if Y is a twisted double $D(W) \cup_f D(W)$, then the Whitehead torsion vanishes [MS, proof of Thm. 3.1]. Since $\dim W + 1$ is even, it follows that τ is nontrivial for many choices of G (*e.g.*, most odd order cyclic groups), and in all such cases one has examples of the desired type.

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Department of Mathematics, Tulane University, New Orleans, Louisiana 70118
E-mail address: `kwasik@math.tulane.edu`

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
Current address, Reinhard Schultz: Department of Mathematics, University of California, Riverside, California 92521
E-mail address: `schultz@math.ucr.edu`