PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1545–1555
S 0002-9939(99)04674-2
Article electronically published on January 29, 1999

THE MODULI SPACE OF $SU(3)$-FLAT CONNECTIONS AND THE FUSION RULES

MASATO HAYASHI

(Communicated by Ronald A. Fintushel)

Abstract. The aim of this paper is to determine the existence condition of the moduli space of $SU(3)$-flat connections on 3-holed 2-sphere D, the so-called pair of pants, and to study its relationship to the $\widehat{\mathfrak{sl}}(3;\mathbb{C})$ fusion rules. The existence condition can be expressed by a system of inequalities with the entries of highest weights with respect to the fundamental weights. This gives a necessary condition for the fusion coefficents to be nontrivial. We also find that the fusion coefficient of a triplet of extremal highest weights equals one. This can be considered a quantum counterpart of the PRV-conjecture.

1. Introduction

Let $G = SU(3)$. Then the set of conjugacy classes of G can be identified with a triangular domain Δ in $\mathfrak{t}$, the Lie algebra of a fixed maximal torus T of G. Fixing a triplet $\Theta = (\alpha, \beta, \gamma) \in \Delta^3$, we consider the moduli space $\mathcal{M}_{D,\Theta}$ of $SU(3)$-flat connections on D associated to Θ. Our main result is the following; the moduli space $\mathcal{M}_{D,\Theta}$ is not empty if and only if the entries of α, β and γ satisfy all the following 18 inequalities:

$$\begin{cases}\alpha_{\sigma(1)} + \beta_{\sigma(1)} + \gamma_{\sigma(3)} \geq 0, & \alpha_{\sigma(1)} + \beta_{\sigma(2)} + \gamma_{\sigma(2)} \geq 0,\\ \alpha_{\sigma(1)} + \beta_{\sigma(3)} + \gamma_{\sigma(3)} \leq 0, & \alpha_{\sigma(2)} + \beta_{\sigma(2)} + \gamma_{\sigma(3)} \leq 0,\end{cases} \tag{1.1}$$

$$\alpha_{\sigma(1)} + \beta_{\sigma(1)} + \gamma_{\sigma(2)} \leq 2\pi, \qquad \alpha_{\sigma(2)} + \beta_{\sigma(3)} + \gamma_{\sigma(3)} \geq -2\pi. \tag{1.2}$$

Here $\sigma \in \mathbb{Z}/3\mathbb{Z}$ acts on the index set $\{i, j, k\}$ of the left hand side $\alpha_i + \beta_j + \gamma_k$ of each inequality as cyclic renumbering (Theorem 3.3).

This condition is obtained by studying the map $\overline{F}_{\alpha\beta} : T\backslash G/T \to \Delta$ which is defined from the equation specifying $\mathcal{M}_{D,\Theta}$. This map is closely related to the moment map, and has a polygonal image denoted by $Q = Q(\alpha, \beta)$. The above condition (1.1), (1.2) is equivalent to the one that $\gamma^* = (-\gamma_3, -\gamma_2, -\gamma_1) \in \Delta$ is included in $Q \subset \Delta$. By investigating the inverse image $\overline{F}^{-1}_{\alpha\beta}(\gamma^*)$, we also determine the topological type of $\mathcal{M}_{D,\Theta}$: this moduli space is homeomorphic to S^2 if γ^* is located in the interior of Q, or one point if γ^* on the boundary of Q (Theorem 3.4). It was already shown by [B] that this moduli space is either empty, a point

Received by the editors May 13, 1996 and, in revised form, September 3, 1997.
1991 *Mathematics Subject Classification.* Primary 58D29, 57M05; Secondary 17B81, 17B10.
Key words and phrases. Representations of the fundamental group of a surface, Bruhat decomposition, fusion rules, PRV-conjecture.

or S^2 under the assumption that the space of stable bundles coincides with that of semistable bundles, whereas our result does not require this assumption.

Let $P_+(3;K)$ be the set of level K dominant integral weights of affine Lie algebra $\widehat{\mathfrak{sl}}(3;\mathbb{C})$ for a fixed positive integer K called the level, and Δ^* be the triangular region, the so-called alcove, defined as the convex hull spanned by $P_+(3;K)$. Constructing a natural isomorphism $f:\Delta \xrightarrow{\cong} \Delta^*$, we set $\lambda = f(\alpha),\ \mu = f(\beta)$ and $\nu = f(\gamma)$ and denote the image of $Q = Q(\alpha,\beta)$ by $Q^* = Q^*(\lambda,\mu)$ under f. Then the condition given in Theorem 3.3 is translated into the analogous results in terms of $\lambda,\ \mu$ and ν. On the other hand, according to [KMSW], the $\widehat{\mathfrak{sl}}(3;\mathbb{C})$ fusion rules are completely determined by the Berenstein-Zelevinsky triangles (see (4.1)). Using this description, it is proved that the fusion coefficient $N_{\lambda\mu}^{\nu^*} = 0$ unless $\lambda,\ \mu$ and ν satisfy the condition This means that, if the fusion coefficient $N_{\lambda\mu\nu} = N_{\lambda\mu}^{\nu^*}$ does not vanish, then the moduli space $\mathcal{M}_{D,\Theta}$ associated to the triplet $\Theta = \big(f^{-1}(\lambda),\ f^{-1}(\mu),\ f^{-1}(\nu)\big) \in \Delta^3$ is not empty. It confirms the correspondence between the space of conformal blocks and the space of generalized theta functions (in case of $G = SU(3)$), which was suggested by Witten ([W]) and was proved by [F], [BL] and [KNR]. We, furthermore, show that the non-zero fusion coefficient $N_{\lambda\mu}^{\nu}$ corresponding to ν on the boundary of $Q^*(\lambda,\ \mu)$ must be exactly one, which is known as the PRV-conjecture in the classical case ([PRV], [Ku]).

2. The moduli space $\mathcal{M}_{D,\Theta}$ and the map $\overline{\Phi}_{\alpha\beta}$

2.1. **The definition of $\mathcal{M}_{D,\Theta}$ and $\overline{\Phi}_{\alpha\beta}$.** For the standard maximal torus T of $G = SU(3)$ as diagonal matrices, we set

$$\Delta = \big\{(\eta_1,\,\eta_2,\,\eta_3)\ \big|\ \eta_1+\eta_2+\eta_3 = 0,\ \eta_1 \geq \eta_2 \geq \eta_3,\ \eta_1 - \eta_3 \leq 2\pi\big\},$$

which is naturally regarded as a subset of $\mathfrak{t} = \mathrm{Lie}(T)$. Then Δ is identified with the space of conjugacy classes $G/\mathrm{Ad}G$ of G. Fixing a triplet $\Theta = (\alpha,\,\beta,\,\gamma) \in \Delta^3$, we assign these conjugacy classes to each component of the boundary of an oriented 3-holed 2-sphere D and denote them by $C_\alpha,\ C_\beta$ and C_γ respectively. We may also denote by C_η $(\eta = \alpha,\ \beta,\ \gamma)$ the elements of the fundamental group $\pi_1(D,*)$ represented by simple closed curves freely homotopic to C_η based at a fixed point $*$ on D. Then the moduli space treated in this paper is defined by

$$\begin{aligned}\mathcal{M}_{D,\Theta} &= \big\{\rho \in \mathrm{Hom}\big(\pi_1(D,*),G\big)\ \big|\ \rho(C_\eta) \sim e_\eta\ (\eta = \alpha,\ \beta,\ \gamma)\big\}/\mathrm{Ad}G\\ &= \big\{(g_\alpha,\,g_\beta,\,g_\gamma) \in G^3\ \big|\ g_\alpha g_\beta g_\gamma = 1,\ g_\eta \sim e_\eta\ (\eta = \alpha,\ \beta,\ \gamma)\big\}/\mathrm{Ad}G.\end{aligned} \tag{2.1}$$

Here $g_\eta = \rho(C_\eta)$ for $\rho \in \mathrm{Hom}\big(\pi_1(D,*),G\big)$ and $g_\eta \sim e_\eta$ means that g_η is conjugate to $e_\eta = \exp(\eta) \in T$ for $\eta \in \Delta$. It is well known that $\mathcal{M}_{D,\Theta}$ can be identified with the moduli space of smooth flat G-connections on D the conjugacy classes of whose holonomies along the oriented loops C_η are equal to η $(\eta = \alpha,\ \beta,\ \gamma)$. So we call it the moduli space of $SU(3)$-flat connections on D associated to Θ.

The equation $g_\alpha g_\beta g_\gamma = 1$ in (2.1) can be rewritten as $e_\alpha h e_\beta h^{-1} = h' e_\gamma^{-1} {h'}^{-1}$ for some elements $h,\,h' \in G$. For $SU(3)$, the conjugacy class of an element is completely determined by the value of its trace and hence the equation above holds if and only if there exists $h \in G$ satisfying $\mathrm{tr}\big(e_\alpha h e_\beta h^{-1}\big) = \mathrm{tr}\big(e_\gamma^{-1}\big)$. So, for two fixed conjugacy classes $\alpha,\ \beta \in \Delta$, we define a map $\widetilde{\Phi}_{\alpha\beta}: G \to \mathbb{C}$ by

$$\widetilde{\Phi}_{\alpha\beta}(h) = \mathrm{tr}\big(e_\alpha h e_\beta h^{-1}\big). \tag{2.2}$$

Let $\Delta' = \mathrm{tr}\big(\exp(\Delta)\big) \subset \mathbb{C}$, then Δ' is the target space of $\widetilde{\Phi}_{\alpha\beta}$ a priori and a connected and simply connected region enclosed by the cycloid $\partial\Delta' = \big\{2e^{\sqrt{-1}\theta} + e^{-2\sqrt{-1}\theta} \in \mathbb{C} \,|\, 0 \leq \theta \leq 2\pi\big\}$. The composition $\Delta \xrightarrow{\exp} T \xrightarrow{\mathrm{tr}} \Delta'$ is a homeomorphism and we introduce the map $\widetilde{F}_{\alpha\beta} : G \xrightarrow{\widetilde{\Phi}_{\alpha\beta}} \Delta' \xrightarrow{\cong} \Delta$.

To determine the existence condition of $\mathcal{M}_{D,\Theta}$, we shall investigate the image of this map and describe it in terms of α, β. Because the map (2.2) is invariant under the action of T on G from both sides, it descends to $\overline{\Phi}_{\alpha\beta} : T\backslash G/T \longrightarrow \Delta'$, as does the map $\widetilde{F}_{\alpha\beta}$ to $\overline{F}_{\alpha\beta} : T\backslash G/T \to \Delta$. Of course, the image of $\overline{F}_{\alpha\beta}$ (resp. $\overline{\Phi}_{\alpha\beta}$) coincides with that of $\widetilde{F}_{\alpha\beta}$ (resp. $\widetilde{\Phi}_{\alpha\beta}$).

2.2. **The description of $T\backslash G/T$ via the Bruhat decomposition.** The main purpose of section 2 is to describe the image $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ and to determine the inverse image of a point in $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$. To carry this out, the topological type of the set of quotient singularities of $T\backslash G/T$ plays an important role because it almost coincides with the quotient image of the critical point set of $\Phi_{\alpha\beta} : G/T \to \Delta'$. We describe it by making use of the Bruhat decomposition of $G_{\mathbb{C}}$ (see [S] for a full accoount).

Let $G_{\mathbb{C}}$, $H = T_{\mathbb{C}}$ be the complexification of G, T respectively and B be a Borel subgroup containing H. The Bruhat decomposition of $G_{\mathbb{C}}$ gives rise to a cell decomposition of the flag manifold $G/T \approx G_{\mathbb{C}}/B = \amalg_{w\in W} B\dot{w}B/B$. Here W is the Weyl group with respect to H, and $\dot{w}$ is a representative of $w \in W$ in N, the normalizer of H in $G_{\mathbb{C}}$. We denote the set of positive roots by R^+, the set of simple roots by Σ, and the set of generators of W corresponding to Σ by S. Noticing that Bruhat cells depend on the choice of S (or Σ), we denote each Bruhat cell $B\dot{w}B/B = N\dot{w}B/B$ by $X^S(w)$.

The left action of T on $G_{\mathbb{C}}/B$ is not free. In order to describe all the sets of singularities, we consider the left action of W on $T\backslash G_{\mathbb{C}}/B$ descending from the natural left action of W on $G_{\mathbb{C}}/B$. In particular, the natural projection $\pi : G_{\mathbb{C}}/B \to T\backslash G_{\mathbb{C}}/B$ is equivariant with respect to the action of W. By this action, $w_1 \in W$ translates each Bruhat cell $X^S(w)$ to

$$w_1 X^S(w) = \exp\Big(\bigoplus_{\alpha\in R^-(w)} \mathfrak{g}_{w_1(\alpha)}\Big)(\dot{\overline{w_1 w}})B \,/\, B, \tag{2.3}$$

where $R^-(w) = \{\alpha \in R^+ \,|\, -w(\alpha) \in R^+\}$. We denote (2.3) by $X^S_{w_1}(w)$ and call it a w_1-shifted Bruhat cell centered at $\dot{\overline{w_1 w}} \in G_{\mathbb{C}}/B$. Using the shifted Bruhat cells, we define the singular 1-cell connecting $\overline{w}$ and $\overline{ws}$ in $T\backslash G/T \approx T\backslash G_{\mathbb{C}}/B$ by

$$e^1(w,\, ws) = \overline{T\backslash X^S_w(s)}$$

for $w \in W$ and $s \in S$. Here $\overline{w} = T\dot{w}B \in T\backslash G_{\mathbb{C}}/B$. Regardless of the choice of S, $s \in S$ and $w \in W$, $X^S_w(s)$ is isomorphic to the complex plane $\mathbb{C}$ and T acts on it as the ordinary action of $S^1 = \{z \in \mathbb{C} \,\big|\, |z| = 1\}$. Hence each singular 1-cell is homeomorphic to the closed interval $[0, 1]$. We can show that the union of singular 1-cells coincides with the quotient image of the part on which the action of T is not free. We denote it by $\mathcal{S} \subset T\backslash G/T$ and call it the singular locus of $T\backslash G/T$.

Consider the graph $\Gamma(\mathfrak{S}_3)$ with a vertex for each element of $\mathfrak{S}_3(\cong W)$ and an edge connecting the vertices corresponding to w, $w' \in \mathfrak{S}_3$ whenever w and w' satisfy

$$w' = ws \tag{2.4}$$

for a transposition $s \in \mathfrak{S}_3$. It is also proved from the description (2.3) that any two singular 1-cells intersect only at their end points and therefore $\mathcal{S}$ is homeomorphic to $\Gamma(\mathfrak{S}_3)$.

We also define a higher dimensional analogue of a singular 1-cell. For $w \in W$ and $S = \{s_i, s_j\}$, the singular 2-cell determined by the ordered quadruplet $\{w, ws_i, ws_j, ws_is_j\}$ is defined as

$$e^2(w, ws_i, ws_j, ws_is_j) = \overline{T\backslash X_w^S(s_is_j)}.$$

The following properties hold for these singular 2-cells.

Lemma 2.1. *For each cycle consisting of four singular 1-cells in $\mathcal{S}$, there exists a singular 2-cell whose boundary is a union of the four singular 1-cells. In particular, each singular 2-cell is homeomorphic to the product of two closed intervals. In this correspondence, points in the singular locus $\mathcal{S}$ are mapped to vertices of cones.*

Proof. The four vertices in such a cycle can always be written as $\{w, ws_i, ws_j, ws_is_j\}$ with an appropriate choice of $S = \{s_i, s_j\}$ and $w \in W$. Owing to the W-equivariance of $\pi : G_{\mathbb{C}}/B \to T\backslash G_{\mathbb{C}}/B$, we have only to check the case for the quadruplet $\{\mathbf{1}, s_i, s_j, s_is_j\}$. It is easy to see that $X_{\mathbf{1}}^S(s_is_j) \approx \mathbb{C}^2$ and $T\backslash X_{\mathbf{1}}^S(s_is_j) \approx [0, 1)^2$ whose boundary is a union of $e^1(s_i, s_is_j) \setminus \{\overline{s_i}\}$ and $e^1(s_j, s_is_j) \setminus \{\overline{s_j}\}$ connected at $\overline{s_is_j}$. On the other hand, according to the closure relation for the Bruhat decomposition,

$$T\backslash \left(\overline{X_{\mathbf{1}}^S(s_is_j)} \setminus X_{\mathbf{1}}^S(s_is_j)\right) = T\backslash \left(X_{\mathbf{1}}^S(s_i) \cup_{X_{\mathbf{1}}^S(\mathbf{1})} X_{\mathbf{1}}^S(s_j)\right) = e^1(\mathbf{1}, s_i) \cup_{\overline{\mathbf{1}}} e^1(\mathbf{1}, s_j).$$

So $\partial e^2(\mathbf{1}, s_i, s_j, s_is_j)$ is the cycle chosen at the beginning and $e^2(\mathbf{1}, s_i, s_j, s_is_j) \approx [0, 1] \times [0, 1]$. $\square$

We conclude this subsection with a description of the local topology around the singular locus $\mathcal{S}$ in $T\backslash G/T$. This is a key lemma in determining the image of the map $\overline{\Phi}_{\alpha\beta} : T\backslash G/T \to \Delta'$ and the fiber $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ above z in its image.

Lemma 2.2. *For an arbitrary point $\overline{x} \in \mathcal{S}$ which does not coincide with $\overline{w} \in \mathcal{S}$ $(w \in W)$, there is a neighborhood $N_{T\backslash G/T}(\overline{x})$ which is homeomorphic to the product of an open interval and a cone on the 2-dimensional sphere S^2. In this correspondence, points in the singular locus $\mathcal{S}$ are mapped to vertices of cones.*

Proof. Note that any shifted 1-dimensional Bruhat cell is included in a shifted big cell. Again, due to the W-equivariance of the natural projection π, it suffices to show the assertion for the special case such as $X_{\mathbf{1}}^S(s_3)$ and $X_{\mathbf{1}}^{S'}(s_3)$ where $S = \{s_1, s_2\}$ and $S' = \{s_2, s_3\}$. Then we can choose as a coordinate for these cells

$$U = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{pmatrix} \right\} \cong X_{\mathbf{1}}^S(s_3), \qquad V = \left\{ \begin{pmatrix} 1 & & z_3 \\ & 1 & \\ & & 1 \end{pmatrix} \right\} \cong X_{\mathbf{1}}^{S'}(s_3),$$

on which T acts by the adjoint action. Choosing a point in V such that $z_3 = x \in \mathbb{R}_{>0}$ as a lift $\widetilde{x} \in X_{\mathbf{1}}^{S'}(s_3)$ of $\overline{x} \in T\backslash X_{\mathbf{1}}^{S'}(s_3)$, we take a neighborhood $U_{\widetilde{x}}$ of $\widetilde{x}$ as follows:

$$U_{\widetilde{x}} = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ & 1 & z_2 \\ & & 1 \end{pmatrix} \in U \;\middle|\; \begin{matrix} |z_1|^2 + |z_2|^2 < r^2 \\ x - \varepsilon < |z_3| < x + \varepsilon \end{matrix} \right\},$$

where r and ε are sufficiently small positive constants. The quotient space $T\backslash U_{\widetilde{x}}$ gives the neighborhood $N_{T\backslash G/T}(\overline{x})$. Other cases are proved similarly. $\square$

2.3. **The image of the map $\overline{\Phi}_{\alpha\beta}$.** It is an easy exercise to deduce the next lemma from the explicit description (2.2) of $\widetilde{\Phi}_{\alpha\beta}(h)$ with matrices because it depends only on the absolute values of the entries of h. For $w \in W$, set $z_w = \mathrm{tr}\big(\exp(\alpha + w(\beta))\big) = \overline{\Phi}_{\alpha\beta}(\overline{w})$.

Lemma 2.3. *Let $\widetilde{P}$ be the convex hull in $\mathbb{C}$ spanned by $\{z_w\}_{w\in W}$. Then the image of the map $\overline{\Phi}_{\alpha\beta}$ is contained in the intersection of $\widetilde{P}$ and Δ'.*

Let $l_{ww'}$ $(w,\, w' \in W)$ be the image of $e^1(w,\, w')$ under the map $\overline{\Phi}_{\alpha\beta}$. It is easy to see that $l_{ww'}$ is a segment connecting z_w and $z_{w'}$ in $\Delta' \subset \mathbb{C}$ and that $e^1(w,\, w')$ is homeomorphically projected onto $l_{ww'}$ by $\overline{\Phi}_{\alpha\beta}$ if z_w and $z_{w'}$ are distinct. We denote by P the connected and simply connected region enclosed by $\overline{\Phi}_{\alpha\beta}(\mathcal{S}) = \bigcup_{\{w,\, w'\}} l_{ww'}$, where $\{w,\, w'\}$ ranges over all the pairs of the elements of $W \cong \mathfrak{S}_3$ satisfying (2.4). By virtue of Lemma 2.1, P is contained in $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$. In short, $P \subset \mathrm{Im}\overline{\Phi}_{\alpha\beta} \subset \widetilde{P} \cap \Delta'$. It is obvious that $\mathrm{Im}\overline{\Phi}_{\alpha\beta} = P = \widetilde{P}$ if P is convex. If P is not convex, namely P has a vertex where the corresponding internal angle of P is greater than π, P and $\widetilde{P}$ do not coincide and $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ cannot be determined from this information alone. We say that such a vertex is located at the concavity of P.

The image $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ depends on the choice of $\alpha,\ \beta \in \Delta$. There are six z_w's and $\mathrm{Im}\overline{\Phi}_{\alpha\beta} = P = \widetilde{P}$ is generally (but not always) a hexagon. Especially α and β located on the boundary $\partial\Delta$ strongly affect $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$. Since the expected existence condition of $\mathcal{M}_{D,\Theta}$ must be symmetric in α, β and γ, we have only to study the case where both α and β are located on $\partial\Delta$ and the one where neither of them is on $\partial\Delta$. In the former case, we can easily show that $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ forms a segment one of whose ends is on $\partial\Delta'$, which is included in the case $\mathrm{Im}\overline{\Phi}_{\alpha\beta} = P = \widetilde{P}$. In the following, we assume that neither α nor β is on the boundary $\partial\Delta$ and call $\alpha,\ \beta$ generic for this case. The next lemma specifies the boundary of $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$.

Lemma 2.4. *Let $\alpha,\ \beta$ be generic. The set of critical values of the map $\Phi_{\alpha\beta}$: $G/T \to \Delta'$ consists of $\bigcup_{\{w,\, w'\}} l_{ww'}$ where $\{w,\, w'\}$ ranges over all the pairs of the elements of $W \cong \mathfrak{S}_3$ satisfying (2.4), and a part of $\partial\Delta'$ if the image $Im\Phi_{\alpha\beta}$ has an intersection with $\partial\Delta'$. In particular, $\mathcal{S}$ is just the quotient image of the critical point set corresponding to $\bigcup_{\{w,\, w'\}} l_{ww'}$.*

Proof. For $h \in G$ and the vector field $\underline{X}$ on G/T determined by $X \in \mathfrak{g}$,

$$\begin{aligned}(d\Phi_{\alpha\beta})_{[h]}\,(\underline{X}) &= \mathrm{tr}\Big(e_\beta h^{-1} e_\alpha h\big(\mathrm{id} - \mathrm{Ad}(e_\beta)\big)(X)\Big)\\ &= 2\sqrt{-1}\Big(\big(\mathrm{Im}\sum_{i<j} y_{ij}\bar{g}_{i1}g_{j1}\big)\big(e^{\sqrt{-1}\theta_1} - e^{\sqrt{-1}\theta_3}\big)\\ &\qquad + \big(\mathrm{Im}\sum_{i<j} y_{ij}\bar{g}_{i2}g_{j2}\big)\big(e^{\sqrt{-1}\theta_2} - e^{\sqrt{-1}\theta_3}\big)\Big),\end{aligned}$$

where $g = (g_{ij})$ is a certain element of G such that $e_\beta h^{-1} e_\alpha h = g e_\theta g^{-1}$ for $\theta \in \Delta$ and $\big(\mathrm{id} - \mathrm{Ad}(e_\beta)\big)(X) = (y_{ij})$. There are two cases where $\mathrm{rank}_{\mathbb{R}}\,(d\Phi_{\alpha\beta})_{[h]} < 2$. An easy calculation shows that either $\mathrm{Im}\sum y_{ij}\bar{g}_{i1}g_{j1}$ or $\mathrm{Im}\sum y_{ij}\bar{g}_{i2}g_{j2}$ vanish if and only if the quotient image of h is contained in a singular 1-cell, whereas $(e^{\sqrt{-1}\theta_1} - e^{\sqrt{-1}\theta_3})\overline{(e^{\sqrt{-1}\theta_2} - e^{\sqrt{-1}\theta_3})}$ is a real number if and only if $\mathrm{tr} e_\theta = \overline{\Phi}_{\alpha\beta}([h]) \in \partial\Delta'$. $\square$

Finally, we conclude with the following proposition concerning the image $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$.

Proposition 2.5. *If P is convex, then the image $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ coincides with P. If P is not convex, $\partial\Delta'$ comes into contact with the two boundary edges of P emanating from the vertex at the concavity of P and the image $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ coincides with the union of P and the domain enclosed by the boundary of P at the concavity and $\partial\Delta'$.*

Proof. In the following, the quotient image of the critical points (resp. regular points) of $\Phi_{\alpha\beta} : G/T \to \Delta'$ are also called the critical points (resp. regular points).

First, it is easy to show from (2.2) that the inverse image $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ of a point z on the boundary $\partial\big(\mathrm{Im}\overline{\Phi}_{\alpha\beta}\big)$ is always one point in $T\backslash G/T$. The restriction of $\overline{\Phi}_{\alpha\beta}$ to a sufficiently small neighborhood $N = N_{T\backslash G/T}(\overline{x})$ of Lemma 2.2 is a projection to a neighborhood of $l_{ww'}$ because it maps the only critical point set $e^1(w, w')$ homeomorphically onto $l_{ww'}$. If we take $\overline{x}$ from $e^1(w, w')$ corresponding to $l_{ww'}$ in the intersection $\partial P \cap \partial\widetilde{P}$ which is a part of $\partial\big(\mathrm{Im}\overline{\Phi}_{\alpha\beta}\big)$, N is projected into P so that the vertices of cones are mapped to the boundary. So the inverse image of a point $z \in P$ near the boundary $\partial P \cap \partial\widetilde{P} = l_{ww'}$ is homeomorphic to S^2.

The interior of P is separated into some regions by $l_{ww'}$'s. If a neighborhood $N = N_{T\backslash G/T}(\overline{x})$ of Lemma 2.2 corresponding to $l_{ww'}$ lying in the interior of P were mapped to the one side of $l_{ww'}$, there had to exist other critical points than $\mathcal{S}$ in the interior of P because $T\backslash G/T$ is connected, which is a contradiction to Lemma 2.4. Hence the restriction $\overline{\Phi}_{\alpha\beta}\big|_N$ is a natural projection locally and $\overline{\Phi}_{\alpha\beta}^{-1}(z) \cap N$ is always homeomorphic to 2-disk D^2 for z near $l_{ww'}$. Therefore the topological type of $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ does not change when $z \in \mathrm{Im}\overline{\Phi}_{\alpha\beta}$ goes across such $l_{ww'}$'s and is always homeomorphic to S^2 for interior points z of $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$, in particular it is connected.

There are two different types for the vertex z at the concavity of P. If it is a crossing point of $l_{ww'}$'s, there exists a regular point in $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ because the critical point set is discrete in it (recall $\mathcal{S} \approx \Gamma(\mathfrak{S}_3)$) and the fiber is connected. If $z = z_w$ for some $w \in W$, we can choose a point in $T\backslash G/T$ which is not contained in $\mathcal{S}$ and is mapped to z by using an explicit expression (2.2) with matrices. In any case, we have a regular point in $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ for the vertex z at the concavity. By the inverse function theorem, the vertex at the concavity can not be a boundary point and, with the help of Lemma 2.4, $\partial\Delta'$ must come into contact with boundary edges emanating from z, and Proposition 2.5 is concluded. $\square$

As a corollary of the proof above, we also obtain the following.

Lemma 2.6. *Let α, β be generic. If $z \in \Delta'$ is a point on the boundary of $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$, then the inverse image $\overline{\Phi}_{\alpha\beta}^{-1}(z)$ is one point in $T\backslash G/T$. The inverse image of an interior point of $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ is homeomorphic to S^2.*

3. The existence condition and the topological type of $\mathcal{M}_{D,\Theta}$

In order to describe the image of the map $\overline{F}_{\alpha\beta}$, we study the homeomorphism $j : \Delta' \to \Delta$ which is given by the inverse of the composition $\Delta \xrightarrow{\exp} T \xrightarrow{\mathrm{tr}} \Delta'$. Since $\mathrm{Im}\overline{\Phi}_{\alpha\beta}$ is a connected and simply connected region in Δ', we have only to study the image of its boundary. To describe concretely the image of $\mathrm{tr} e_\eta = e^{\sqrt{-1}\eta_1{}'} + e^{\sqrt{-1}\eta_2{}'} + e^{\sqrt{-1}\eta_3{}'} \in \Delta'$ where $\eta_1{}' + \eta_2{}' + \eta_3{}' = 0$ under the map j, we consider an action on $\mathfrak{t}$ ($\supset \Delta$) of a group isomorphic to the affine Weyl group $\widehat{W}$ of the affine Lie algebra $\widehat{\mathfrak{sl}}(3;\mathbb{C})$. Since Δ is thought to be a fundamental region with respect to the action of this group (also denoted by $\widehat{W}$), we have the natural projection $\widehat{p} : \mathfrak{t} \longrightarrow \mathfrak{t}/\widehat{W} \cong \Delta$. Then $j(\eta') = \widehat{p}(\eta') \in \Delta$ where $\eta' = (\eta_1{}', \eta_2{}', \eta_3{}') \in \mathfrak{t}$. So we can describe the image of $\overline{w} \in T\backslash G/T$ ($w \in W$) under the map $\overline{F}_{\alpha\beta}$ as

$$\overline{F}_{\alpha\beta}(\overline{w}) = \widehat{p}\big(\alpha + w(\beta)\big). \tag{3.1}$$

The image of $l_{ww'}$'s under the map j is described by the following lemma.

Lemma 3.1. *The image of $l_{ww'}$ via the map j is a possibly broken segment connecting $\overline{F}_{\alpha\beta}(\overline{w})$ and $\overline{F}_{\alpha\beta}(\overline{w'})$, each piece of which is vertical to one of the edges of $\partial\Delta$. Moreover, $j(l_{ww'})$ is broken only at the intersection point with $\partial\Delta$.*

Proof. For simplicity, we consider the case $w = \mathbf{1}$, $w' = (12) \in W \cong \mathfrak{S}_3$. The locus drawn by $\eta = \big(\eta_1, \eta_2, \eta_3\big) \in \Delta$ satisfying $\eta_1 + \eta_2 + (\alpha_3 + \beta_3) \equiv 0 \pmod{2\pi}$ is a segment connecting $\widehat{p}\,(\alpha + w(\beta))$ and $\widehat{p}\,(\alpha + w'(\beta))$, each piece of which is vertical to one of the edges of $\partial\Delta$ and is possibly broken at the intersection points with $\partial\Delta$. On the other hand, from the expression

$$j^{-1}(\eta) = \mathrm{tr} e_\eta = 2e^{-\frac{\sqrt{-1}}{2}(\alpha_3+\beta_3)} \cos\frac{\eta_1 - \eta_2}{2} + e^{\sqrt{-1}(\alpha_3+\beta_3)},$$

we see that the image of the locus drawn by η is also a segment connecting $\overline{\Phi}_{\alpha\beta}(\overline{w})$ and $\overline{\Phi}_{\alpha\beta}(\overline{w'})$, which is just $l_{ww'}$. Since j is a homeomorphism, the assertion follows. $\square$

Owing to Proposition 2.5 and Lemma 3.1, we can determine $\mathrm{Im}\overline{F}_{\alpha\beta}$.

Proposition 3.2. *Let $V = V(\alpha, \beta)$ be the set of points given by (3.1), then the image $\mathrm{Im}\overline{F}_{\alpha\beta}$ is the convex polygon whose points $\eta = (\eta_1, \eta_2, \eta_3) \in \Delta$ satisfy*

$$\min\big\{\zeta_i \;\big|\; (\zeta_1, \zeta_2, \zeta_3) \in V\big\} \le \eta_i \le \max\big\{\zeta_i \;\big|\; (\zeta_1, \zeta_2, \zeta_3) \in V\big\} \qquad (i = 1, 2, 3). \tag{3.2}$$

For fixed α, β, we denote by $Q = Q(\alpha, \beta)$ the convex polygon in Δ defined by the inequalities (3.2). Following the procedure described before Lemma 3.1, we rewrite this result with the entries of α, β. Since the condition that $\mathcal{M}_{D,\Theta}$ is non-empty is equivalent to the one that $\gamma^* = j^{-1}\big(\mathrm{tr}\, e_\gamma^{-1}\big) = \big(-\gamma_3, -\gamma_2, -\gamma_1\big) \in \Delta$ is included in Q, we arrive at the first main result of this paper.

Theorem 3.3. *For $\Theta = \big(\alpha, \beta, \gamma\big) \in \Delta^3$, we write $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Then $\mathcal{M}_{D,\Theta}$ is not empty if and only if α, β and γ satisfy all the following 18 inequalities :*

$$\begin{cases} \alpha_{\sigma(1)} + \beta_{\sigma(1)} + \gamma_{\sigma(3)} \ge 0, & \alpha_{\sigma(1)} + \beta_{\sigma(2)} + \gamma_{\sigma(2)} \ge 0, \\ \alpha_{\sigma(1)} + \beta_{\sigma(3)} + \gamma_{\sigma(3)} \le 0, & \alpha_{\sigma(2)} + \beta_{\sigma(2)} + \gamma_{\sigma(3)} \le 0, \end{cases} \tag{3.3}$$

$$\alpha_{\sigma(1)} + \beta_{\sigma(1)} + \gamma_{\sigma(2)} \le 2\pi, \qquad \alpha_{\sigma(2)} + \beta_{\sigma(3)} + \gamma_{\sigma(3)} \ge -2\pi. \tag{3.4}$$

Here $\sigma \in \mathbb{Z}/3\mathbb{Z}$ acts on the index set $\big\{i, j, k\big\}$ of the left-hand side $\alpha_i + \beta_j + \gamma_k$ of each inequality as cyclic renumbering.

Let $\alpha,\ \beta$ be generic. From the definition (2.1), we can choose as a complete representative system of $\mathcal{M}_{D,\Theta}$

$$T\backslash\{h \in G \mid \operatorname{tr}\left(e_\alpha h e_\beta h^{-1}\right) = \operatorname{tr}\left(e_\gamma^{-1}\right)\}/T,$$

which is nothing but $\overline{F}_{\alpha\beta}^{-1}(\gamma^*) \subset T\backslash G/T$. For the case neither α nor β is generic, it is easy to see that $\overline{F}_{\alpha\beta}^{-1}(\gamma^*)$ is always one point. By adding the result of Lemma 2.6 to this consequence, we obtain the second theorem of this paper.

Theorem 3.4. *Let a triplet $\Theta = (\alpha,\ \beta,\ \gamma) \in \Delta^3$ satisfy all the inequalities in Theorem 3.3, then the moduli space $\mathcal{M}_{D,\Theta}$ is homeomorphic to S^2 or one point:*

$$\mathcal{M}_{D,\Theta} \approx \begin{cases} S^2 & \textit{(if none of the equalities in (3.3), (3.4) holds)}, \\ \textit{one point} & \textit{(otherwise)}. \end{cases}$$

4. The existence condition of $\mathcal{M}_{D,\Theta}$ and the fusion rules

4.1. **The relation between the existence condition of $\mathcal{M}_{D,\Theta}$ and the fusion rules.** First we make a quick digression to the fusion algebras and the explicit description of the $\widehat{\mathfrak{sl}}(3;\mathbb{C})$ fusion rules due to [KMSW]. For more details about the fusion algebras, the reader is referred to [K], [Fu], [GN]. For the sake of simplicity, we shall restrict our subject to the $A_{n-1}^{(1)}$ $(n \geq 2)$ affine Lie algebras in what follows.

Let $P_+(n)$ be the set of the dominant integral weights of $\mathfrak{g} = \mathfrak{sl}(n;\mathbb{C})$. For a fixed positive integer K called the level, we define $P_+(n;K)$ as the set of $\lambda = [\lambda_1, \cdots, \lambda_{n-1}] \in P_+(n)$ satisfying $\sum \lambda_i \leq K$. The fusion algebra $R_{n,K}$ is a free $\mathbb{Z}$-module with basis $\lambda \in P_+(n;K)$. For the decomposition of its product

$$\lambda \cdot \mu = \sum_\nu N_{\lambda\mu}^\nu \nu,$$

it is known that the fusion coefficient $N_{\lambda\mu}^\nu$ is a non-negative integer. Denoting by ν^* the highest weight of the dual representation of V_ν, we set $N_{\lambda\mu\nu} = N_{\lambda\mu}^{\nu^*}$. If $N_{\lambda\mu\nu} \neq 0$, then we say that $\lambda,\ \mu$ and ν satisfy the fusion rules. In case of $G = SU(2)$, they are well-known as the quantum Clebsch-Gordan conditions ([TK]). In case of $G = SU(3)$, they are described by [KMSW] in a completely combinatorial way.

Berenstein and Zelevinsky show that the (classical) Littlewood-Richardson coefficient $\overline{N}_{\lambda\mu}^{\nu^*} = \overline{N}_{\lambda\mu\nu}$ in the representation ring R_3 of $\mathfrak{sl}(3;\mathbb{C})$ coincides with the number of triangles which one can construct according to the following rules:

$$\tag{4.1} \begin{array}{ccc} & a_1 & \\ & a_2\ a_9 & \\ & a_3 \quad\ a_8 & \\ & a_4\ a_5\ a_6\ a_7 & \end{array} \quad \text{such that} \quad \begin{aligned} a_1 + a_2 &= \lambda_1, \\ a_3 + a_4 &= \lambda_2, \\ a_4 + a_5 &= \mu_1, \\ a_6 + a_7 &= \mu_2, \\ a_7 + a_8 &= \nu_1, \\ a_9 + a_1 &= \nu_2, \end{aligned} \qquad \begin{aligned} a_2 + a_3 &= a_6 + a_8, \\ a_3 + a_5 &= a_9 + a_8, \\ a_5 + a_6 &= a_2 + a_9, \end{aligned}$$

with non-negative integer a_i $(i = 1, \cdots, 9)$. They are called the Berenstein-Zelevinsky (BZ) triangles. In [KMSW] the threshold level $k_0(\mathfrak{X})$ of a BZ triangle $\mathfrak{X}$ is defined by $k_0(\mathfrak{X}) = \max\{a_1+\mu_1+\mu_2,\ a_4+\nu_1+\nu_2,\ a_7+\lambda_1+\lambda_2\}$ and it is shown that the fusion coefficient $N_{\lambda\mu\nu}$ is obtained as the number of BZ triangles whose threshold level is not greater than the level $K:$ $N_{\lambda\mu\nu} = \sharp\{\mathfrak{X} \in BZ(\lambda,\ \mu,\ \nu) \mid k_0(\mathfrak{X}) \leq K\}$.

Now we establish the correspondence between the existence condition of $\mathcal{M}_{D,\Theta}$ and the fusion rules by identifying $\Delta \subset \mathfrak{t}$ with the alcove $\Delta^* \subset \mathfrak{h}^*$ which is the convex hull spanned by $P_+(3; K)$. This identification is given by a $\widehat{W}$-equivariant natural linear map $f : \mathfrak{t} \to \mathfrak{h}^*$, where $\mathfrak{h}^*$ is a dual Cartan subalgebra of $\mathfrak{g}$. Let us denote the image of $Q = Q(\alpha, \beta) \subset \Delta$ by $Q^* = Q^*(\lambda, \mu) \subset \Delta^*$ for $\lambda = f(\alpha)$, $\mu = f(\beta)$ under the isomorphism f. We can rewrite (3.3), (3.4) in terms of the entries of λ, μ and ν with this correspondence. If the fusion coefficient $N_{\lambda\mu\nu}$ is non-zero, there exists a BZ triangle $\mathfrak{X}$ satisfying $k_0(\mathfrak{X}) \leq K$. With the aid of this fact, we can show that all the 18 inequalities rewritten in the entries of λ, μ and ν hold, e.g.

$$\begin{aligned} 2\lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - 2\nu_2 &= 3(a_2 + a_4) \geq 0, \\ 2\lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 + \nu_2 &= 3(a_1 + a_4 + a_5 + a_6) \leq 3K, \end{aligned} \tag{4.2}$$

etc. As a consequence, we see that if the fusion coefficient $N_{\lambda\mu\nu} \neq 0$, then the moduli space $\mathcal{M}_{D,\Theta} \neq \emptyset$ where $\Theta = \big(f^{-1}(\lambda), f^{-1}(\mu), f^{-1}(\nu)\big) \in \Delta^3$. For $\xi = [\xi_1, \xi_2] \in \mathfrak{h}^*$ with (not necessarily non-negative) integer ξ_i $(i = 1, 2)$ in the weight lattice of $\mathfrak{g}$, we use the notation $\widehat{\xi} = \widehat{p}(\xi)$ (see (3.1)). Let $V^* = V^*(\lambda, \mu) = \big\{\widehat{\lambda + w(\mu)} \;\big|\; w \in W\big\}$. We state this consequence in a more representation theoretical style.

Theorem 4.1. *The fusion coefficient $N_{\lambda\mu}^{\nu} = 0$ unless $\nu = [\nu_1, \nu_2]$ satisfies one of the inequalities*

$$\begin{aligned} \min\big\{2\xi_1 + \xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\} &\leq 2\nu_1 + \nu_2 \leq \max\big\{2\xi_1 + \xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\}, \\ \min\big\{-\xi_1 + \xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\} &\leq -\nu_1 + \nu_2 \leq \max\big\{-\xi_1 + \xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\}, \\ \min\big\{\xi_1 + 2\xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\} &\leq \nu_1 + 2\nu_2 \leq \max\big\{\xi_1 + 2\xi_2 \;\big|\; [\xi_1, \xi_2] \in V^*\big\}. \end{aligned}$$

4.2. **Some comments on the $\widehat{\mathfrak{sl}}(3;\mathbb{C})$ fusion rules.** Theorem 4.1 only gives a necessary condition for the fusion coefficients to be nontrivial, and the converse is not true. However, it seems to give the best possible condition in the following sense. In all the examples that we have checked, the extremal highest weights (i.e. those located at the vertices of Q^*) always occure as a factor of the product $\lambda \cdot \mu$ in the fusion algebra. Let us cite an example.

Example 4.2. Let $\lambda = [2, 1]$, $\mu = [1, 2]$ and $K = 5$, then

$$[2, 1]\widehat{\otimes}[1, 2] = [4, 1] \oplus [1, 4] \oplus 2[2, 2] \oplus [3, 0] \oplus [0, 3] \oplus 2[1, 1] \oplus [0, 0]$$

(see Figure 1). On the one hand, $Q^* = Q^*(\lambda, \mu)$ is given by

$$Q^* = \left\{ [\xi_1, \xi_2] \in P_+(3; K) \;\middle|\; \begin{array}{c} 0 \leq 2\xi_1 + \xi_2 \leq 9 \\ -3 \leq -\xi_1 + \xi_2 \leq 3 \\ 0 \leq \xi_1 + 2\xi_2 \leq 9 \end{array} \right\},$$

which is a pentagon with its vertices $[4, 1]$, $[1, 4]$, $[3, 0]$, $[0, 3]$ and $[0, 0]$.

As Witten suggested via a path integral argument ([W]), the fusion coefficient $N_{\lambda\mu\nu}$ coincides with the dimension of the space of the holomorphic sections of a certain line bundle on the moduli space $\mathcal{M}_{D,\Theta}$ associated to $\Theta = \big(f^{-1}(\lambda), f^{-1}(\mu), f^{-1}(\nu)\big)$ (the Verlinde formula). This is proved by [F], [BL], [KNR], and Theorem 4.1 can be regarded as an elementary confirmation of this correspondence in case of $SU(3)$. If we apply their results to the moduli space of semistable parabolic bundles on $\mathbb{C}P^1 \setminus \big\{0, 1, \infty\big\}$ with weights α, β, γ at 0, 1, ∞

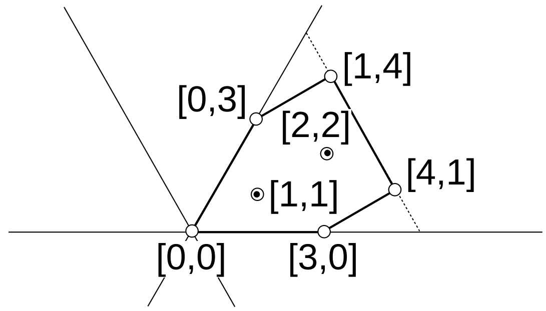

FIGURE 1. The components in the decomposition of $[2,\,1]\widehat{\otimes}[1,\,2]$

respectively —–this moduli space is identified with our moduli space $\mathcal{M}_{D,\Theta}$ ([MS]), we obtain the following result since $\mathcal{M}_{D,\Theta}$ is just a point.

Theorem 4.3. *The non-zero fusion coefficient $N_{\lambda\mu}^{\nu}$ corresponding to the highest weight ν located on the boundary of $Q^*(\lambda,\,\mu)$ (i.e. ν satisfying one of equalities in the inequalities in Theorem 4.1) must be exactly one.*

For the classical case of a finite dimensional Lie algebra $\mathfrak{g}$, the analogous result is known as the PRV-conjecture ([PRV]), which was established by [Ku]. It states that, in the tensor product $V_\lambda \otimes V_\mu$ of two finite dimensional irreducible highest weight $\mathfrak{g}$-module, the irreducible highest weight module $V_{\overline{\lambda+w(\mu)}}$ corresponding to extremal highest weight $\overline{\lambda+w(\mu)}$ $(w \in W)$ occurs with multiplicity exactly one. We therefore pose the following problem which, given Theorem 4.3, is the quantum counterpart of the PRV-conjecture.

Problem 4.4. *The extremal highest weight ν at each vertex of $Q^*(\lambda,\,\mu)$ will always appear in the decomposition of the product $\lambda\cdot\mu$ in the fusion algebra. Furthermore, ν satisfies the condition of Theorem 4.1, then $N_{\lambda\mu}^{\nu} \neq 0$ if and only if the left-hand side of (4.2) for $\lambda,\ \mu$ and ν^* is divisible by 3.*

REFERENCES

[B] H. U. Boden, *Unitary representations of Brieskorn spheres*, Duke Math. **75(No.1)** (1994), 193–220. MR **95f:**57033

[BL] A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta function*, Comm. Math. Phys. **164** (1994), 385–419. MR **95k:**14011

[BZ] A. D. Berenstein and A. Z. Zelevinsky, Cornell preprint-technical report **90-60** (1990).

[F] G. Faltings, *A proof of the Verlinde formula*, J. Alg. Geom. **3** (1994), 347–374. MR **95j:**14013

[Fu] J. Fuchs, *Affine Lie algebras and quantum groups*, Cambridge, 1995. MR **96a:**17018

[GN] F. M. Goodman and T. Nakanishi, *Fusion algebras in integrable systems in two dimensions*, Phys. Lett. **B262** (1991), 259–264. MR **92i:**81281

[K] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge, 1990. MR **92k:**17038

[KMSW] A. N. Kirillov, P. Mathieu, D. Sénéchal and M. A. Walton, *Can fusion coefficients be calculated from the depth rule?*, Nuclear Phys. **B391** (1993), 651–674. MR **94b:**81110

[KNR] S. Kumar, M. S. Narasimhan and A. Ramanathan, *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. **300** (1994), 41–75. MR **96e:**14011

[Ku] S. Kumar, *Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture*, Invent. Math. **93** (1988), 117–130. MR **89j:**17009

[MS] V. B. Mehta and C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structure*, Math. Ann. **248** (1980), 205–239. MR **81i:**14010

[PRV] K. R. Parthasarathy, R. Ranga Rao and V. S. Varadarajan, *Representations of complex semi-simple Lie groups and Lie algebras*, Ann.of Math. **85** (1967), 383–429. MR **37:**1526

[S] T. A. Springer, *Linear algebraic groups*, Birkhäuser, 1981. MR **84i:**20002

[TK] A. Tsuchiya and Y. Kanie, *Vertex operators in conformal field theory on P^1 and monodromy representations of braid group*, Adv. Stu. Pure Math. **85** (1988), 297–372. MR **89m:**81166

[W] E. Witten, *The Verlinde algebra and the cohomology of the Grassmannian*, IASSNS-HEP-93/41. CMP 96:03

Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo

E-mail address: hayashim@ms318sun.ms.u-tokyo.ac.jp