

AN ELEMENTARY PROOF OF THE PRINCIPLE OF LOCAL REFLEXIVITY

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ABSTRACT. We give an elementary proof of the principle of local reflexivity.

We use only elementary functional analysis to give a simple and short proof of the version of the “principle of local reflexivity” proved in [2], which is an improvement of the original version given in [3]. Other short proofs can be found in [1] and [4]. We use standard notation for Banach spaces. By X , X^* and X^{**} , we denote a real or complex Banach space, its first dual and its second dual respectively; we identify X with the canonical copy of X contained in X^{**} ; given a Banach space Y , we write $B_Y := \{y : \|y\| \leq 1\}$ and $S_Y := \{y : \|y\| = 1\}$; given a subset A of X , $\bar{A}^{\sigma(X^{**}, X^*)}$ stands for the *weak**-closure of A in X^{**} and $\text{int}A$ is the norm interior of A ; an operator is a continuous linear function; given $\varepsilon > 0$, an ε -isometry $T : E \rightarrow Y$ is an operator for which $1 - \varepsilon \leq \|Tx\| \leq 1 + \varepsilon$ for all $x \in S_E$.

We only require the following Lemma 1. We omit its proof, which is an easy exercise based on the separation Hahn-Banach theorem.

Lemma 1. *Let $T : X \rightarrow Y$ be an operator, $z \in \text{int}B_{X^{**}}$ and $y \in Y$ such that $\|T^{**}z - y\| < \varepsilon$. Then we have that $z \in \bar{L}^{\sigma(X^{**}, X^*)}$, where $L := \{x \in B_X : \|Tx - y\| < \varepsilon\}$.*

Theorem 2 (Principle of local reflexivity). *Let $E \subset X^{**}$ and $F \subset X^*$ be finite dimensional subspaces. Given $\varepsilon > 0$ there exists an ε -isometry $T : E \rightarrow X$ such that $T|_{E \cap X} = \text{id}$ and $f(Te) = e(f)$ for all $f \in F$ and all $e \in E$.*

Proof. Let $\dim E = n$ and $\dim E \cap X = n - k$. Let $(y_j, h_j)_{j=1}^n$ be a biorthogonal system in $E \times E^*$ such that $\|y_j\| = 1 - \varepsilon$ and $\text{span}\{y_j\}_{j=k+1}^n = E \cap X$. The identity $\text{id} : E \rightarrow X^{**}$ can be given as $\text{id}(e) = \sum_{j=1}^n h_j(e)y_j$. We shall find v_1, \dots, v_k in X so that the operator $T : E \rightarrow X$ defined by $T(e) := \sum_{j=1}^k h_j(e)v_j + \sum_{j=k+1}^n h_j(e)y_j$ is an ε -isometry. Hence, the condition $T|_{E \cap X} = \text{id}$ will be satisfied automatically.

Let $W := X^k$ endowed with the norm $\|(x_j)_{j=1}^k\| = \sup_j \|x_j\|$, and select $0 < \alpha < \min\{2/5, (1 - \varepsilon)^{-1} - 1, \varepsilon(\sum_{j=1}^n \|h_j\|)^{-1}\}$. Fix $\{f_j\}_{j=1}^M$ a basis in F , $\{e_j\}_{j=1}^N$

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an $\alpha/4$ -net in $\text{int}B_E$, and $\{u_j\}_{j=1}^N$ in B_{X^*} so that $\|e\| \leq (1 + \alpha) \sup_{1 \leq j \leq N} |e(u_j)|$ for all $e \in E$. We have that

$$e_j = \sum_{r=1}^n \lambda_r^j y_r, j = 1, \dots, N.$$

Let us write $P := \max_{1 \leq j \leq N} \sum_{r=1}^k |\lambda_r^j|$ and define the set

$$C := \left\{ (x_s)_{s=1}^k \in B_W : \left\| \sum_{s=1}^k \lambda_s^j x_s + \sum_{s=k+1}^n \lambda_s^j y_s \right\| < 1, j = 1, \dots, N \right\}.$$

By the above lemma, we have that $(y_j)_{j=1}^k \in \overline{C}^{\sigma(W^{**}, W^*)}$. Now we set the operator $S : W \longrightarrow \mathbf{R}^{M \cdot k + N \cdot k}$ (or into $\mathbf{C}^{M \cdot k + N \cdot k}$) given by $S((x_s)_{s=1}^k) := (f_i(x_r), u_j(x_s))$ for $1 \leq i \leq M, 1 \leq r \leq k, 1 \leq j \leq N, 1 \leq s \leq k$.

Thus $S^{**}((y_j)_{j=1}^k) \in \overline{S(C)}$. Now, since $\overline{W}^{\sigma(W^{**}, W^*)} = W^{**}$ and $R(S)$ is closed, we have that $R(S) = R(S^{**})$, and then, for $0 < \beta < \min\{1, \varepsilon(2P)^{-1}\}$, we can find $(c_j)_{j=1}^k \in C$ and $(b_j)_{j=1}^k \in \beta B_W$ so that

$$S^{**}((y_j)_{j=1}^k) = S((c_j)_{j=1}^k) + S((b_j)_{j=1}^k).$$

We take $v_j := c_j + b_j$ for $j = 1, \dots, k$ in the definition of T . Thus, we already have the condition $f(Te) = e(f)$ for all $f \in F$ and all $e \in E$. Now, since $\|Te_j\| \leq 1 + \|\sum_{r=1}^k \lambda_r^j b_r\| \leq 1 + \beta P$ for $j = 1, \dots, N$, it is completely straightforward to check that T is an ε -isometry.

□

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