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THE LOCATION OF THE ZEROS OF THE HIGHER ORDER DERIVATIVES OF A POLYNOMIAL

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ABSTRACT. Let $p(z)$ be a complex polynomial of degree n having k zeros in a disk D. We deal with the problem of finding the smallest concentric disk containing $k-l$ zeros of $p^{(l)}(z)$. We obtain some estimates on the radius of this disk in general as well as in the special case, where k zeros in D are isolated from the other zeros of $p(z)$. We indicate an application to the root-finding algorithms.

1. INTRODUCTION

Let us consider the following problem: If k zeros of a polynomial $p(z)$ of degree n $(2 \leq k \leq n)$ lie in a disk D of radius r, what is the smallest concentric disk that contains $k-l$ zeros of $p^{(l)}(z)$ $(1 \leq l \leq k-1)$? Since the problem is scaling and translation invariant, we can assume that the disk D is the closed unit disk $\overline{\Delta} := \{z \in \mathbb{C} : |z| \leq 1\}$. Let $\mathcal{P}_{n,k}$ denote the class of complex polynomials of degree n having exactly k zeros in $\overline{\Delta}$. We define the function $\rho(n,k,l)$, $n \geq k > l$, as follows:

(1.1)
$$\rho(n,k,l) = \sup_{p \in \mathcal{P}_{n,k}} \min \left\{ R > 0 : \overline{D}(0,R) \text{ contains at least } k-l \text{ zeros of } p^{(l)}(z) \right\}.$$

Because of scaling and translation invariance we can conclude that if $\overline{D}(c,r)$ contains k zeros of the polynomial $p(z)$, then $\overline{D}(c, r\rho(n,k,l))$ contains $k-l$ zeros of $p^{(l)}(z)$.

The problem of estimating $\rho(n,k,l)$ has a long history in the case $l=1$. The results listed below can be found in Marden's book ([2]). The Gauss-Lucas Theorem states that $\rho(n,n,1) = 1$. Result $\rho(n,2,1) = \cot(\pi/n)$ is due to Alexander, Kakeya and Szegő. Biernacki proved that

$$\rho(n,n-1,1) \leq (1+1/n)^{1/2} \qquad \text{and} \qquad \rho(n,k,1) \leq \prod_{i=1}^{n-k} [(n+i)/(n-i), \tag{1.2}$$

and Marden showed that

$$\rho(n,k,1) \leq \csc \frac{\pi}{2(n-k+1)}. \tag{1.3}$$

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More recently, Coppersmith and Neff ([1]) proved that, under the condition that k zeros in $\overline{\Delta}$ are centered at 0,

$$\rho(n,k,l) \le 10.477\frac{(n-k+1)(k-l)}{\sqrt{k}} \tag{1.4}$$

and

$$\rho(n,k,k-1) \le C\max\left\{(n-k+1)^{1/2}k^{-1/4}, (n-k+1)k^{-2/3}\right\}. \tag{1.5}$$

One can also impose an additional condition on $p \in \mathcal{P}_{n,k}$ to have zeros in Δ isolated from the other zeros. Following Pan ([3]), we define *the isolation ratio of a polynomial* p *with respect to a disk* D, $I(p,D)$, as:

$$I(p,D(c,r)) := \sup\{\mu > 0 : D(c,\mu r) \text{ contains exactly the same zeros of } p \text{ as } D(c,r)\}. \tag{1.6}$$

We say that a disk D is *f-isolated* if $I(p,D) \ge f$. The isolation ratio is also scaling and translation invariant. We define *the isolation ratio of* p as $I(p) := I(p,\overline{\Delta})$. Renegar ([5]) proved the following result:

(1.7)

$$\text{If } I(p) \ge 15n^3, \text{ then } \rho(n,k,k-1) \le \frac{3n}{2} \text{ and } I\left(p^{(k-1)}, \overline{D}(0,3n/2)\right) \ge \frac{I(p)}{10n^2}.$$

In other words, if k zeros of a polynomial p are well-isolated in some disk, then $p^{(k-1)}$ has an isolated single zero in a larger concentric disk. Using Walsh's Coincidence Theorem ([2]) we improve the bound in (1.7). We also generalize Biernacki's proof of (1.2) and obtain an upper bound for $\rho(n,k,l)$ which is smaller than (1.4) for some special choices of n, k and l.

Results of this type have found applications in constructing low complexity algorithms for finding zeros of polynomials. Smale and Renegar ([7], [5]) derived quantitative criteria for a point to be in the domain of quadratic convergence of Newton's algorithm. The basic idea is that if a zero is isolated from other zeros, then we need only crude approximation to obtain fast convergence. However, if we have an isolated cluster of k zeros, then $p^{(k-1)}$ should have a single zero nearby. In this case, the result like (1.7) allows us to apply Newton's algorithm to $p^{(k-1)}$ with quadratic convergence. To establish initial isolation of clusters of zeros, various global search algorithms are used ([4], [5])

2. Main results

First we prove the following proposition:

Proposition 2.1.

$$\text{If } I(p) \ge 1 + 2\frac{l(n-k)}{k-l+1}, \text{ then } \rho(n,k,l) \le 1 \text{ and } I\left(p^{(l)}\right) \ge \frac{(k-l+1)I(p) - l(n-k)}{k-l+1+l(n-k)}. \tag{2.1}$$

Proof. Let $I(p) = R$ and let $p(z) = f(z)g(z)$ where $f(z)$ is a polynomial of degree k whose zeros lie in $\overline{\Delta}$ and $g(z)$ is a polynomial of degree $n-k$ whose zeros lie in

$\{z : |z| > R\}$. Let $m = \min(n-k, l)$. If z is a solution of the equation

$$p^{(l)}(z) = \sum_{i=0}^{m} \binom{l}{i} g^{(i)}(z) f^{(l-i)}(z) = 0, \tag{2.2}$$

then the Walsh Coincidence Theorem implies that there exist $x_1, |x_1| \leq 1$ and $x_2, |x_2| \geq R$ such that

$$\sum_{i=0}^{m} \binom{l}{i} \left((z-x_2)^{n-k}\right)^{(i)} \left((z-x_1)^{k}\right)^{(l-i)} = 0, \tag{2.3}$$

i.e.

$$\sum_{i=0}^{m} \binom{l}{i} \frac{(n-k)!}{(n-k-i)!}(z-x_2)^{n-k-i} \frac{k!}{(k-l+i)!}(z-x_1)^{k-l+i} = 0, \tag{2.4}$$

i.e.

$$(z-x_1)^{k-l}(z-x_2)^{n-k-m} \sum_{i=0}^{m} \binom{l}{i} \binom{n-l}{n-k-i} (z-x_2)^{m-i}(z-x_1)^{i} = 0. \tag{2.5}$$

Also z must lie in the interval $[x_1, x_2]$ and either $z = x_1$ or $z = x_2$ or z is a zero of the polynomial

$$q(z) = \sum_{i=0}^{m} \binom{l}{i} \binom{n-l}{n-k-i} (z-x_2)^{m-i}(z-x_1)^{i}. \tag{2.6}$$

Let $w = \dfrac{z-x_1}{z-x_2}$ and let

$$h(w) = \sum_{i=0}^{m} \binom{l}{i} \binom{n-l}{n-k-i} w^{i} = \sum_{i=0}^{m} a_i w^i. \tag{2.7}$$

All zeros of $h(w)$ lie on the negative real axis and by the Eneströ̋m-Kakeya Theorem they satisfy the inequality

$$\min_{0\leq i\leq m-1} \frac{a_i}{a_{i+1}} \leq |w| \leq \max_{0\leq i\leq m-1} \frac{a_i}{a_{i+1}}. \tag{2.8}$$

We have that

$$\frac{a_i}{a_{i+1}} = \frac{(i+1)(k-l+i+1)}{(l-i)(n-k-i)}; \tag{2.9}$$

therefore

$$\zeta := \min_{0\leq i\leq m-1} \frac{a_i}{a_{i+1}} = \frac{k-l+1}{l(n-k)} \tag{2.10}$$

and

$$\xi := \max_{0\leq i\leq m-1} \frac{a_i}{a_{i+1}} = \frac{m(m+k-l)}{(l-m+1)(n-k-m+1)}. \tag{2.11}$$

Since all zeros of $h(w)$ lie in the interval $[-\xi, -\zeta]$, all zeros of $q(z)$ lie in the interval $\left[\frac{\zeta x_2 + x_1}{\zeta+1}, \frac{\xi x_2 + x_1}{\xi+1}\right]$. By continuity, $p^{(l)}(z)$ has $k-l$ zeros in $\overline{\Delta}$ if $\left|\frac{\zeta x_2+x_1}{\zeta+1}\right| \geq 1$. Since

$\left|\frac{\zeta x_2+x_1}{\zeta+1}\right| \leq \frac{\zeta R-1}{\zeta+1}$, the above inequality holds if $\frac{\zeta R-1}{\zeta+1} \geq 1$, i.e. if $R \geq \left(1+\frac{2}{\zeta}\right) = \left(1+2\frac{l(n-k)}{k-l+1}\right)$. These are the only zeros of $p^{(l)}(z)$ in the disk

$$D\left(0, \frac{\zeta R-1}{\zeta+1}\right) = D\left(0, \frac{(k-l+1)R-l(n-k)}{k-l+1+l(n-k)}\right). \tag{2.12}$$

□

We use Proposition 2.1 to prove the following result analogous to (1.7).

Theorem 2.2.

$$\textit{If } I(p) \geq \frac{n^2}{4}, \ \textit{then } \rho(n,k,k-1) \leq 1 \ \textit{and } I\left(p^{(k-1)}\right) \geq \frac{4I(p)}{n^2}. \tag{2.13}$$

Proof. If $l = k-1$ and $I(p) \geq 1+(k-1)(n-k)$, then, by Proposition 2.1, $\overline{\Delta}$ contains a zero of $p^{(k-1)}$. Since $1+(k-1)(n-k) \leq k(n-k) \leq n^2/4$, the same is true if $I(p) \geq n^2/4$. Also,

(2.14)
$$I\left(p^{(k-1)}\right) \geq \frac{2I(p)-(k-1)(n-k)}{2+(k-1)(n-k)} > \frac{I(p)}{1+(k-1)(n-k)} \geq \frac{I(p)}{k(n-k)} \geq \frac{4I(p)}{n^2}.$$

□

Remark 2.3. We don't know of any estimates for $\rho(n,k,k-1)$ if the isolation ratio of p is smaller than $n^2/4$. In applications to zero-finding we can use the classical Graeffe process of root-squaring to control the isolation ratio of a polynomial ([3],[6]). If p_0 has zeros $z_1, \ldots, z_n$ and isolation ratio $C > 1$, then the polynomial

$$p_1(z) := p_0(\sqrt{z})p_0(-\sqrt{z}) \tag{2.15}$$

has zeros $z_1^2, \ldots, z_n^2$ and isolation ratio C^2. By iterating this process we obtain polynomial p_m with zeros $z_1^{2^m}, \ldots, z_n^{2m}$ and isolation ratio C^{2^m}.

We can also apply Proposition 2.1 to obtain an upper bound for $\rho(n,k,l)$.

Theorem 2.4.

$$\rho(n,k,l) \leq \frac{n+l}{n-l}\prod_{i=k}^{n-2}\left(1+\frac{2l(n-i)}{i-l+1}\right). \tag{2.16}$$

Proof. Let us assume that the zeros of $p(z)$ are ordered by their moduli, $|z_1| \leq |z_2| \leq \ldots \leq |z_n|$. If $k = n-1$, then $p(z) = f(z)(z-z_n)$, where $f(z)$ is a polynomial with zeros $z_i, |z_i| \leq 1$, $1 \leq i \leq n-1$. We have

$$p^{(l)}(z) = f^{(l)}(z)(z-z_n) + lf^{(l-1)}(z). \tag{2.17}$$

By the Walsh Coincidence Theorem, if z is a zero of $p^{(l)}(z)$ it is also a zero of the equation:

$$((z-\gamma)^{n-1})^{(l)}(z)(z-z_n) + l((z-\gamma)^{n-1})^{(l-1)}(z) = 0 \tag{2.18}$$

where $|\gamma| \leq 1$, i.e.,

$$(z-\gamma)^{n-1-l}((n-l)(z-z_n)+l(z-\gamma)) = 0. \tag{2.19}$$

Therefore $z = \gamma$ or $z = ((n-l)z_n + l\gamma)/n$. Also if $|((n-l)z_n + l\gamma)/n| > 1$, then there are exactly $k-1-l$ zeros of $p^{(l)}(z)$ in $\overline{\Delta}$. This is the case if $|z_n| > \frac{n+l}{n-l}$. Otherwise, by the Gauss-Lucas Theorem, all zeros of $p^{(l)}(z)$ are in the disk $D\left(0, \frac{n+l}{n-l}\right)$. Now let's fix K and suppose that (2.16) holds for $k = n-1 \ldots K-1$. By Proposition 2.1, if $|z_{K+1}| > 1 + 2\frac{l(n-K)}{K-l+1}$, then there are exactly $K-l$ zeros of $p^{(l)}(z)$ in $\overline{\Delta}$. Otherwise there are $K+1$ zeros of $p(z)$ in the disk $D\left(0, 1+2\frac{l(n-K)}{K-l+1}\right)$ and by induction there are at least $K-l$ zeros of $p^{(l)}(z)$ in $D(0,R)$, where

$$R \leq \frac{n+l}{n-l} \prod_{i=K}^{n-2} \left(1 + \frac{2l(n-i)}{i-l+1}\right). \tag{2.20}$$

$\square$

Inequality (2.16) can be better in some cases than (1.4), namely if k and l are close to n. In the most interesting case $l = k-1$, Coppersmith and Neff gave a lower bound on $\rho(n,k,k-1)$ and also conjectured that if $k > n/2$, then $\rho(n,k,k-1)$ is bounded by a constant.

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