# HAMILTONIAN STATIONARY NORMAL BUNDLES OF SURFACES IN R ${ }^{3}$ 

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#### Abstract

A surface in $\mathbf{R}^{3}$ has Hamiltonian stationary normal bundle if and only if it is either minimal, a part of a round sphere, or a part of a cone with vertex angle $\pi / 2$.


## 0. Introduction

Let $N$ be an $n$-dimensional Kähler manifold with the Kähler form $\Omega$. A Lagrangian submanifold $M$ in $N$ is an $n$-dimensional submanifold in $N$ such that $\left.\Omega\right|_{M}=0$. A Lagrangian submanifold in a Kähler manifold is called Hamiltonian stationary if it is a critical point of the volume functional for all Hamiltonian deformations (cf. [2]). Of course, any minimal Lagrangian submanifold is Hamiltonian stationary. In fact, a Lagrangian submanifold with parallel mean curvature is Hamiltonian stationary (cf. Section 1).

On the other hand, let $M$ be a submanifold in $\mathbf{R}^{n}$. The normal bundle $T^{\perp} M$ of $M$ may be naturally immersed in $\mathbf{R}^{n} \times \mathbf{R}^{n}$ by the immersion $\psi: T^{\perp} M \rightarrow \mathbf{R}^{n} \times \mathbf{R}^{n}$ defined by $\psi\left(\nu_{x}\right)=\left(x, \nu_{x}\right)$. We consider the complex structure $J$ on $C^{n}=\mathbf{R}^{n} \times \mathbf{R}^{n}$ defined by $J(X, Y)=(-Y, X)$. Then it is known that $\psi\left(T^{\perp} M\right)$ is a Lagrangian submanifold in $C^{n}=\mathbf{R}^{n} \times \mathbf{R}^{n}$ (cf. [1, III.3.C]).

So it is natural to ask which submanifolds in $\mathbf{R}^{n}$ have Hamiltonian stationary normal bundles. In this paper we discuss the case of surfaces in $\mathbf{R}^{3}$.

Theorem. Let $S$ be a surface in $\mathbf{R}^{3}$. Then $\psi\left(T^{\perp} S\right)$ is Hamiltonian stationary if and only if $S$ is either minimal, a part of a round sphere, or a part of a cone with vertex angle $\pi / 2$.

In the proof of the theorem, we can find that the normal bundles of a round sphere and a cone with vertex angle $\pi / 2$ in $\mathbf{R}^{3}$ have non-parallel mean curvature.

Remark. Harvey and Lawson determined submanifolds in $\mathbf{R}^{n}$ with minimal normal bundles (see [1, III. Th. 3.11, Prop. 2.17]). In particular, they showed that a surface $S$ in $\mathbf{R}^{n}$ has minimal normal bundle if and only if $S$ is minimal.

The author wishes to thank the referee for comments.

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## 1. Preliminaries

Let $N$ be an $n$-dimensional Kähler manifold with the complex structure $J$ and the Kähler metric $g$. The Kähler form $\Omega$ of $N$ is defined by $\Omega(X, Y)=g(J X, Y)$ for $X, Y \in T_{x} N$. Let $M$ be a Lagrangian submanifold in $N$, that is, an $n$-dimensional submanifold in $N$ such that $\left.\Omega\right|_{M}=0$. Then $J$ becomes a bijection between $T_{x} M$ and $T_{x}^{\perp} M$ for $x \in M$.

A normal vector field $V$ along $M$ is called a Hamiltonian variation if $V=$ $J(\operatorname{grad}(f))$ for some compactly supported function $f$ on $M$. Let $i: M \rightarrow N$ be the inclusion map. A compactly supported deformation $\phi_{t}: M \rightarrow N(-\varepsilon<t<$ $\varepsilon, \phi_{0}=i$ ) of $M$ is called a Hamiltonian deformation if its variation vector field is a Hamiltonian variation. We say that $M$ is Hamiltonian stationary if it satisfies

$$
\left.\frac{d}{d t}\right|_{t=0} \operatorname{vol}\left(\phi_{t}(M)\right)=0
$$

for all Hamiltonian deformations $\phi_{t}$. The Euler-Lagrange equation is given as follows:

Proposition (cf. [2]). Let $N$ be a Kähler manifold with the complex structure J. A Lagrangian submanifold $M$ in $N$ is Hamiltonian stationary if and only if its mean curvature vector $H$ satisfies $\operatorname{div}(J H)=0$ on $M$.

From this proposition, we can see that a Lagrangian submanifold with parallel mean curvature in a Kähler manifold is Hamiltonian stationary.

## 2. Proof of the Theorem

Let $x=x\left(t_{1}, t_{2}\right),\left(t_{1}, t_{2}\right) \in D$, be a local parametrization of $S$, where $D$ is an open domain on $\mathbf{R}^{2}$. Let

$$
\mathrm{I}=E\left(d t_{1}\right)^{2}+2 F d t_{1} d t_{2}+G\left(d t_{2}\right)^{2}
$$

and

$$
\mathrm{II}=L\left(d t_{1}\right)^{2}+2 M d t_{1} d t_{2}+N\left(d t_{2}\right)^{2}
$$

be the first and second fundamental forms of $S$, respectively. As our argument is local in nature, we may assume that either $S$ has no umbilic points or $S$ is totally umbilic. In the case where $S$ has no umbilic points, we may choose the local parametrization so that the parameter curves are lines of curvature, and $F=M=$ 0 . It is possible to choose the local parametrization such that $F=M=0$ also in the case where $S$ is totally umbilic. So we assume that $F=M=0$ in the following.

The principal curvatures $a$ and $b$ of $S$ are given by $a=L / E$ and $b=N / G$. By the Codazzi equation, we have

$$
\begin{equation*}
\frac{\partial a}{\partial t_{2}}=\frac{b-a}{2 E} \frac{\partial E}{\partial t_{2}}, \quad \frac{\partial b}{\partial t_{1}}=\frac{a-b}{2 G} \frac{\partial G}{\partial t_{1}} \tag{1}
\end{equation*}
$$

Let $\nu$ denote the unit normal vector along $S$. Let $f: D \times R \rightarrow \mathbf{R}^{3} \times \mathbf{R}^{3}$ be defined by

$$
f\left(t_{1}, t_{2}, t_{3}\right)=\left(x\left(t_{1}, t_{2}\right), t_{3} \nu\left(t_{1}, t_{2}\right)\right)
$$

which is the parametrization of the immersion $\psi: T^{\perp} S \rightarrow \mathbf{R}^{3} \times \mathbf{R}^{3}$ in the introduction.

Set

$$
\begin{gather*}
e_{1}=\left(E\left(1+t_{3}^{2} a^{2}\right)\right)^{-1 / 2} \frac{\partial}{\partial t_{1}} \\
e_{2}=\left(G\left(1+t_{3}^{2} b^{2}\right)\right)^{-1 / 2} \frac{\partial}{\partial t_{2}}, \quad e_{3}=\frac{\partial}{\partial t_{3}} \tag{2}
\end{gather*}
$$

Then we have

$$
\begin{gather*}
f_{*} e_{1}=\left(E\left(1+t_{3}^{2} a^{2}\right)\right)^{-1 / 2}\left(\frac{\partial x}{\partial t_{1}},-t_{3} a \frac{\partial x}{\partial t_{1}}\right) \\
f_{*} e_{2}=\left(G\left(1+t_{3}^{2} b^{2}\right)\right)^{-1 / 2}\left(\frac{\partial x}{\partial t_{2}},-t_{3} b \frac{\partial x}{\partial t_{2}}\right), \quad f_{*} e_{3}=(0, \nu) \tag{3}
\end{gather*}
$$

which are orthonormal. So $\left\{e_{1}, e_{2}, e_{3}\right\}$ is an orthonormal frame on $D \times R$ with respect to the metric induced by $f$.

Let $J$ be the complex structure on $C^{3}=\mathbf{R}^{3} \times \mathbf{R}^{3}$ defined by $J(X, Y)=(-Y, X)$. Set

$$
\begin{align*}
e_{4}=J\left(f_{*} e_{1}\right) & =\left(E\left(1+t_{3}^{2} a^{2}\right)\right)^{-1 / 2}\left(t_{3} a \frac{\partial x}{\partial t_{1}}, \frac{\partial x}{\partial t_{1}}\right) \\
e_{5}=J\left(f_{*} e_{2}\right) & =\left(G\left(1+t_{3}^{2} b^{2}\right)\right)^{-1 / 2}\left(t_{3} b \frac{\partial x}{\partial t_{2}}, \frac{\partial x}{\partial t_{2}}\right)  \tag{4}\\
& e_{6}=J\left(f_{*} e_{3}\right)=(-\nu, 0)
\end{align*}
$$

Then $\left\{e_{4}, e_{5}, e_{6}\right\}$ is a normal orthonormal frame.
The second fundamental form $h_{i j}^{\alpha}$ of $f$ is defined by $h_{i j}^{\alpha}=\left\langle e_{i}\left(f_{*} e_{j}\right), e_{\alpha}\right\rangle$ for $1 \leq i$, $j \leq 3,4 \leq \alpha \leq 6$. Using (2), (3), (4) and (1), we get

$$
\begin{aligned}
h_{11}^{4} & =-t_{3} E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-3 / 2} \frac{\partial a}{\partial t_{1}}, \\
h_{22}^{4} & =-t_{3} E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-1}\left(\frac{a-b}{2 G} \frac{\partial G}{\partial t_{1}}\right) \\
& =-t_{3} E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-1} \frac{\partial b}{\partial t_{1}}, \\
h_{11}^{5} & =-t_{3} G^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1}\left(1+t_{3}^{2} b^{2}\right)^{-1 / 2}\left(\frac{b-a}{2 E} \frac{\partial E}{\partial t_{2}}\right) \\
& =-t_{3} G^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1}\left(1+t_{3}^{2} b^{2}\right)^{-1 / 2} \frac{\partial a}{\partial t_{2}}, \\
h_{22}^{5} & =-t_{3} G^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-3 / 2} \frac{\partial b}{\partial t_{2}}, \\
h_{11}^{6} & =-a\left(1+t_{3}^{2} a^{2}\right)^{-1}, \quad h_{22}^{6}=-b\left(1+t_{3}^{2} b^{2}\right)^{-1}, \\
h_{33}^{4} & =h_{33}^{5}=h_{33}^{6}=0 .
\end{aligned}
$$

Set

$$
\begin{align*}
P= & E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-3 / 2} \frac{\partial a}{\partial t_{1}} \\
& +E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-1} \frac{\partial b}{\partial t_{1}}, \\
Q= & G^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1}\left(1+t_{3}^{2} b^{2}\right)^{-1 / 2} \frac{\partial a}{\partial t_{2}}  \tag{5}\\
& +G^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-3 / 2} \frac{\partial b}{\partial t_{2}}, \\
R= & a\left(1+t_{3}^{2} a^{2}\right)^{-1}+b\left(1+t_{3}^{2} b^{2}\right)^{-1} .
\end{align*}
$$

Then the mean curvature vector $H$ of $f$ is given by

$$
\begin{equation*}
H=-\left(t_{3} P e_{4}+t_{3} Q e_{5}+R e_{6}\right) . \tag{6}
\end{equation*}
$$

By (6) and (4),

$$
J H=t_{3} P\left(f_{*} e_{1}\right)+t_{3} Q\left(f_{*} e_{2}\right)+R\left(f_{*} e_{3}\right),
$$

and we have

$$
\begin{align*}
\operatorname{div}(J H)= & \sum_{i=1}^{3}\left\langle\left(e_{i}(J H)\right)^{T}, f_{*} e_{i}\right\rangle=\sum_{i=1}^{3}\left\langle e_{i}(J H), f_{*} e_{i}\right\rangle \\
= & t_{3}\left(e_{1} P\right)+t_{3} Q\left\langle e_{1}\left(f_{*} e_{2}\right), f_{*} e_{1}\right\rangle+R\left\langle e_{1}\left(f_{*} e_{3}\right), f_{*} e_{1}\right\rangle  \tag{7}\\
& +t_{3} P\left\langle e_{2}\left(f_{*} e_{1}\right), f_{*} e_{2}\right\rangle+t_{3}\left(e_{2} Q\right)+R\left\langle e_{2}\left(f_{*} e_{3}\right), f_{*} e_{2}\right\rangle \\
& +t_{3} P\left\langle e_{3}\left(f_{*} e_{1}\right), f_{*} e_{3}\right\rangle+t_{3} Q\left\langle e_{3}\left(f_{*} e_{2}\right), f_{*} e_{3}\right\rangle+e_{3} R .
\end{align*}
$$

Here $\left(e_{i}(J H)\right)^{T}$ denotes the tangential part of $e_{i}(J H)$, and the seventh and eighth terms of the right hand side are zero.
(i) If $S$ is minimal, then $a+b=0$. By (5) and (6) we have $H=0$, and $\operatorname{div}(J H)=0$.
(ii) If $S$ is a part of a round sphere, then $a=b$ which is constant. By (5), $P=Q=0$ and $R=2 a\left(1+t_{3}^{2} a^{2}\right)^{-1}$. Using (7), (2) and (3), we can see that $\operatorname{div}(J H)=0$.
(iii) In what follows, we assume that $S$ is neither minimal nor a part of a round sphere. Then we may assume that $a^{2} \neq b^{2}$.
(iii) We assume further that $S$ is non-flat. Then we may assume that $a \neq 0$ and $b \neq 0$. Using (7), (5), (2) and (3), we can find that

$$
\begin{align*}
& \left(1+t_{3}^{2} a^{2}\right)^{3}\left(1+t_{3}^{2} b^{2}\right)^{3} \operatorname{div}(J H) \\
& \quad=t_{3}\left\{\left(1+t_{3}^{2} a^{2}\right) A_{1}-\frac{3 a}{E}\left(\frac{\partial a}{\partial t_{1}}\right)^{2} t_{3}^{2}\left(1+t_{3}^{2} b^{2}\right)^{3}\right\}  \tag{8}\\
& \quad=t_{3}\left\{\left(1+t_{3}^{2} b^{2}\right) B_{1}-\frac{3 b}{G}\left(\frac{\partial b}{\partial t_{2}}\right)^{2} t_{3}^{2}\left(1+t_{3}^{2} a^{2}\right)^{3}\right\} \tag{9}
\end{align*}
$$

for some functions $A_{1}, B_{1}$ on $D \times R$, which are polynomial with respect to $t_{3}$.
If $\operatorname{div}(J H)=0$, then (8) and (9) are identically zero, which are true also as polynomials for $t_{3} \in C$. So by letting $t_{3}=\sqrt{-1} / a$ and $\sqrt{-1} / b$ in (8) and (9)
respectively, we get

$$
\begin{equation*}
\frac{\partial a}{\partial t_{1}}=\frac{\partial b}{\partial t_{2}}=0 \tag{10}
\end{equation*}
$$

By (5) and (10),

$$
\begin{align*}
& P=E^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-1} \frac{\partial b}{\partial t_{1}} \\
& Q=G^{-1 / 2}\left(1+t_{3}^{2} a^{2}\right)^{-1}\left(1+t_{3}^{2} b^{2}\right)^{-1 / 2} \frac{\partial a}{\partial t_{2}} \tag{11}
\end{align*}
$$

Again by using (7), (11), (5), (2), (3) and (1), we have
$\left(1+t_{3}^{2} a^{2}\right)^{2}\left(1+t_{3}^{2} b^{2}\right)^{2} \operatorname{div}(J H)$

$$
\begin{align*}
& =t_{3}\left(1+t_{3}^{2} a^{2}\right) A_{2}+t_{3}\left(1+t_{3}^{2} b^{2}\right)\left\{\frac{1+a(2 a-b) t_{3}^{2}}{(b-a) G}\left(\frac{\partial a}{\partial t_{2}}\right)^{2}-a^{3}\left(1+t_{3}^{2} b^{2}\right)\right\}  \tag{12}\\
& =t_{3}\left(1+t_{3}^{2} b^{2}\right) B_{2}+t_{3}\left(1+t_{3}^{2} a^{2}\right)\left\{\frac{1+b(2 b-a) t_{3}^{2}}{(a-b) E}\left(\frac{\partial b}{\partial t_{1}}\right)^{2}-b^{3}\left(1+t_{3}^{2} a^{2}\right)\right\} \tag{13}
\end{align*}
$$

for some functions $A_{2}$ and $B_{2}$ on $D \times R$, which are polynomials with respect to $t_{3}$. As we assume that $\operatorname{div}(J H)=0,(12)$ and (13) are identically zero, also as polynomials for $t_{3} \in C$. So by letting $t_{3}=\sqrt{-1} / a$ and $\sqrt{-1} / b$ in (12) and (13) respectively, we get

$$
\begin{align*}
G & =\frac{1}{a^{2}\left(a^{2}-b^{2}\right)}\left(\frac{\partial a}{\partial t_{2}}\right)^{2} \neq 0  \tag{14}\\
E & =\frac{1}{b^{2}\left(b^{2}-a^{2}\right)}\left(\frac{\partial b}{\partial t_{1}}\right)^{2} \neq 0 \tag{15}
\end{align*}
$$

Inserting (14) into (1), and noting (10), (15), we have a contradiction.
Thus in this case (iii) $)_{1}, \operatorname{div}(J H)$ cannot be identically zero.
(iii) $)_{2}$ We assume that $S$ is flat. Then we may assume that $a=0$ and $b \neq 0$ on $S$. By (5),

$$
\begin{gather*}
P=E^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-1} \frac{\partial b}{\partial t_{1}} \\
Q=G^{-1 / 2}\left(1+t_{3}^{2} b^{2}\right)^{-3 / 2} \frac{\partial b}{\partial t_{2}}, \quad R=b\left(1+t_{3}^{2} b^{2}\right)^{-1} \tag{16}
\end{gather*}
$$

Using (7), (16), (2) and (3), we have

$$
\begin{equation*}
\left(1+t_{3}^{2} b^{2}\right)^{3} \operatorname{div}(J H)=t_{3}\left\{\left(1+t_{3}^{2} b^{2}\right) B_{3}-\frac{3 b}{G}\left(\frac{\partial b}{\partial t_{2}}\right)^{2} t_{3}^{2}\right\} \tag{17}
\end{equation*}
$$

for some function $B_{3}$ on $D \times R$, which is a polynomial with respect to $t_{3}$.

If $\operatorname{div}(J H)=0$, then the right side of (17) is identically zero, also as a polynomial for $t_{3} \in C$. Thus we have $\partial b / \partial t_{2}=0$. So $Q=0$ by (16). Again by using (7), (16), $(2),(3)$ and (1), we get

$$
\begin{align*}
& \operatorname{div}(J H)=t_{3} E^{-1}\left[b^{-1}\left(1+t_{3}^{2} b^{2}\right)^{-2}\left\{\left(\frac{\partial b}{\partial t_{1}}\right)^{2}-E b^{4}\right\}\right.  \tag{18}\\
& \left.+\left(1+t_{3}^{2} b^{2}\right)^{-1}\left\{\frac{\partial^{2} b}{\partial t_{1}^{2}}-\frac{2}{b}\left(\frac{\partial b}{\partial t_{1}}\right)^{2}-\frac{1}{2 E} \frac{\partial E}{\partial t_{1}} \frac{\partial b}{\partial t_{1}}\right\}\right] .
\end{align*}
$$

As we assume that $\operatorname{div}(J H)=0$, by (18), we have

$$
\begin{gather*}
E=\frac{1}{b^{4}}\left(\frac{\partial b}{\partial t_{1}}\right)^{2} \neq 0  \tag{19}\\
\frac{\partial^{2} b}{\partial t_{1}^{2}}-\frac{2}{b}\left(\frac{\partial b}{\partial t_{1}}\right)^{2}-\frac{1}{2 E} \frac{\partial E}{\partial t_{1}} \frac{\partial b}{\partial t_{1}}=0 \tag{20}
\end{gather*}
$$

where we note that (19) implies (20) automatically. By (1) and that $b$ depends only on $t_{1}$,

$$
G=\frac{c}{b^{2}}, \quad N=\frac{c}{b}
$$

for some positive function $c$ depending only on $t_{2}$. Thus we have

$$
\mathrm{I}=\frac{1}{b^{4}}\left(\frac{\partial b}{\partial t_{1}}\right)^{2}\left(d t_{1}\right)^{2}+\frac{c}{b^{2}}\left(d t_{2}\right)^{2}, \quad \mathrm{II}=\frac{c}{b}\left(d t_{2}\right)^{2}
$$

Noting that $c>0$ and $\partial b / \partial t_{1} \neq 0$, we change the parameters as follows:

$$
\tilde{t}_{1}=\frac{1}{b}, \quad \tilde{t}_{2}=\int \sqrt{c} d t_{2}
$$

Then we have

$$
\begin{equation*}
\mathrm{I}=\left(d \tilde{t}_{1}\right)^{2}+\tilde{t}_{1}^{2}\left(d \tilde{t}_{2}\right)^{2}, \quad \mathrm{II}=\tilde{t}_{1}\left(d \tilde{t}_{2}\right)^{2} \tag{21}
\end{equation*}
$$

The cone parametrized by

$$
\begin{equation*}
x\left(t_{1}, t_{2}\right)=\frac{1}{\sqrt{2}}\left(t_{1} \cos \left(\sqrt{2} t_{2}\right), t_{1} \sin \left(\sqrt{2} t_{2}\right), t_{1}\right) \tag{22}
\end{equation*}
$$

has the first and second fundamental forms (21) without tilde, with respect to the suitable unit normal vector field. So by the fundamental theorem, up to congruence, $S$ must be a part of the cone with vertex angle $\pi / 2$.

Conversely, let $S$ be a part of a cone with vertex angle $\pi / 2$. Then we may assume that $S$ is parametrized by (22), and (21) without tilde is valid. With respect to this parametrization, (18) may be applied, where $E=1$ and $b=1 / t_{1}$. By (18) we can see that $S$ has Hamiltonian stationary normal bundle.

Thus the proof is complete.

## References

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[^0]:    Received by the editors June 16, 1997 and, in revised form, August 19, 1997.
    1991 Mathematics Subject Classification. Primary 53C42; Secondary 53A05.

