

POSITIVE DIFFERENTIALS, THETA FUNCTIONS AND HARDY H^2 KERNELS

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ABSTRACT. Let R be a planar regular region whose Schottky double \hat{R} has genus g and set $\hat{T}_0 = \{z \in \mathbb{C}^g | \sqrt{-1}z \in \mathbb{R}^g\}$. For fixed $a \in R$ we determine the range of the function $F(e) = \theta(a - \bar{a} + e)/\theta(e)$ ($e \in \hat{T}_0$) where $\theta(z)$ is the Riemann theta function on \hat{R} . Also we introduce two weighted Hardy spaces to study the problem when the matrix $(\frac{\partial^2 \log F}{\partial z_i \partial \bar{z}_j}(e))$ is positive definite. The proof relies on new theta identities using Fay's trisecants formula.

1. INTRODUCTION

Let R be a planar regular region with n (≥ 2) boundary components $\Gamma_0, \dots, \Gamma_{n-1}$. Its Schottky double \hat{R} is a compact Riemann surface of genus $g = n - 1$ admitting an anti-conformal involution ϕ fixing the boundary ∂R of R . For simplicity we adopt the notation that $\bar{z} = \phi(z)$ for $z \in \hat{R}$ and $\bar{R} = \phi(R)$. The closure of the set S is denoted by $\text{Cl}(S)$.

In 1972 D. A. Hejhal proved for planar regular regions the inequality $C_B(z)^2 < \pi K(z, \bar{z})$ where $C_B(z)$ is the analytic capacity of R at $z \in R$ and $K(z, \bar{w})$ is the Bergman kernel [2]. This inequality was derived from some identity between the Szegő kernel and the Bergman kernel. The key point of the proof was the positive definiteness of the matrix $(\frac{\partial^2 \log \theta}{\partial z_i \partial \bar{z}_j}(0))$.

S. Saitoh considered an analogous problem for Hardy H^2 kernel and posed an open problem, in our context, to prove the negative definiteness of $(\frac{\partial^2 \log \theta}{\partial z_i \partial \bar{z}_j}(e_0))$ [4, p.37]. The constant e_0 is determined from the critical points of Green's function of R whose definition will be given in section 2.

Although we were not able to prove the above conjecture for $n \geq 3$, we show in the last section its relative version such as the matrix $(\frac{\partial^2 \log F}{\partial z_i \partial \bar{z}_j}(e))$ is positive definite for e in some open set in \hat{T}_0 (Theorem 3.2).

By using Hejhal and Fay's results we can easily rewrite many of Widom's results in the context of theta functions. In the next section we treat the extremal value of the function $F(e) = \theta(a - \bar{a} + e)/\theta(e)$ ($e \in \hat{T}_0$) (Theorem 2.1) which is in the author's point of view one of the essences of Widom's paper [5].

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We fix a symmetric canonical homology basis $\{A_i, B_j\}$ ($i, j = 1, \dots, g$) on R such that $B_j = \Gamma_j$ ($j = 1, \dots, g$) and the cycles $\{A_i\}$ ($i = 1, \dots, g$) satisfy the relations in $H_1(\hat{R}, \mathbb{Z})$:

$$\phi(A_i) = -A_i, \quad \phi(B_j) = B_j \quad (i, j = 1, \dots, g).$$

Let u_1, \dots, u_g be the normalized differentials of the first kind on \hat{R} such that $\int_{A_i} u_j = 2\pi\sqrt{-1}\delta_{ij}$ (Kronecker delta); then

$$(1) \quad \phi^* u_j = \overline{u_j} \quad (j = 1, \dots, g).$$

The period matrix τ of \hat{R} is by definition the $g \times g$ matrix $(\int_{B_i} u_j)$ ($i, j = 1, \dots, g$). It is well-known that τ is hermitian with $\operatorname{Re} \tau < 0$. In our case, however, from symmetry (1) we see easily that τ is a real symmetric matrix.

Remark 1.1. Here it should be pointed out that our choice of the canonical homology basis is different from Fay's lecture note [1]: we interchanged the A_i cycles with B_j cycles. Thus some of Fay's formulas must be modified suitably according as the transformation law of theta functions for the change of the homology basis [1, p. 7]. When we treat the Green's function and the Bergman kernel, however, our choice leads to simpler formulas than Fay's, which will be seen later in this paper.

The first order theta function with characteristic $\begin{bmatrix} \delta \\ \varepsilon \end{bmatrix}$ ($\delta, \varepsilon \in \mathbb{R}^g$) is defined by

$$\theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z) = \sum_{m \in \mathbb{Z}^g} \exp\left\{\frac{1}{2}(m + \delta)\tau(m + \delta)^t + (z + 2\pi i\varepsilon)(m + \delta)^t\right\}, \quad z \in \mathbb{C}^g.$$

For $j = 1, \dots, g$, $\theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z)$ satisfies the identities

$$(2) \quad \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z_1, \dots, z_j + 2\pi i, \dots, z_g) = e^{2\pi i\delta_j} \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z)$$

and

$$(3) \quad \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z_1 + \tau_{j1}, \dots, z_g + \tau_{jg}) = e^{-\frac{1}{2}\tau_{jj} - z_j - 2\pi i\varepsilon_j} \theta \begin{bmatrix} \delta \\ \varepsilon \end{bmatrix} (z).$$

Riemann's theta function is denoted by $\theta(z) = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z)$.

The prime-form is given by

$$E(x, y) = \frac{\theta[\alpha](y - x)}{h_\alpha(x)h_\alpha(y)}, \quad x, y \in \hat{R},$$

where α is a non-singular odd half-period and h_α is a "half-order" differential on \hat{R} satisfying $h_\alpha^2(x) = \sum_{j=1}^g \frac{\partial \theta[\alpha]}{\partial z_j}(0)u_j(x)$. For fixed $x \in R$, $E(x, y)$ is a multiplicative $-\frac{1}{2}$ order differential in y with multipliers 1 and $\exp(-\frac{\tau_{jj}}{2} - \int_x^y u_j)$ along the A_j and B_j cycles respectively.

For relevant properties of theta functions and prime-forms used in this paper, our basic reference is the excellent lecture note by J. D. Fay [1].

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2. POSITIVE DIFFERENTIALS AND INEQUALITIES

For any point $p \in \partial R$, a local coordinate w of a neighborhood U of p is called *symmetric* if $w(U \cap \partial R) \subset \mathbb{R}$ and $w(U \cap R) \subset \{\operatorname{Im} z > 0\}$. A differential η on R is called *positive* (resp. *strictly positive*) when η is the restriction of some Abelian differential $\tilde{\eta}$ on \hat{R} such that for any symmetric local coordinate w of a point in ∂R the function $\tilde{\eta}(z)/dw$ is non-negative (resp. positive) on ∂R . For $a \in R$ let P_a be the set of positive differentials on R which is holomorphic except for a simple pole at a . For $a, b \in \hat{R}$ let ω_{a-b} denote the Abelian differential of the third kind on \hat{R} with poles of residue 1 and -1 at a and b respectively. The differential ω_{a-b} is expressed as a logarithmic derivative of the prime-form (cf. [1, p. 17]):

$$(4) \quad \omega_{a-b}(z) = d \log \frac{E(z, a)}{E(z, b)}.$$

If $g(z, a)$ is the Green's function on R , then it is well-known that the differential $dg(z, a) + id^*g(z, a)$ is extended to an Abelian differential of the third kind $\omega_{\bar{a}-a}$ satisfying $i\omega_{\bar{a}-a} \in P_a$. We denote by $\{z_j^*\}_{j=1}^g$ the critical points of $g(\cdot, a)$ which coincide with the zeroes of $\omega_{\bar{a}-a}$ in R .

In connection with the extremal problems on the generalized Tchebycheff polynomials, H. Widom [5] studied multi-valued analytic functions on R and obtained a result on the ranges of some extremal quantities. On the other hand, by using theta functions, J. D. Fay [1] found that the set $\hat{T}_0 = \{z \in \mathbb{C}^g \mid \sqrt{-1}z \in \mathbb{R}^g\}$ gives a parametrization of the set of positive differentials on R with one simple pole.

We shall assume throughout the paper that the divisor of the form $a - \bar{a}$ is always evaluated along a symmetric path of integration. That is, $a - \bar{a} = (\int_C u_j)_{j=1}^g$ where C is a path from \bar{a} to a satisfying $\phi(C) = -C$. This assumption assures us that by (1) we have $a - \bar{a} \in \hat{T}_0$.

We summarize propositions useful in our paper.

Lemma 2.1 (Fay). *The following hold.*

1. $\overline{\theta(x - y - e)} = \theta(\bar{x} - \bar{y} + e)$, for $x, y \in \hat{R}$ and $e \in \hat{T}_0$,
2. $E(\bar{x}, \bar{y}) = \overline{E(x, y)}$, for $x, y \in \hat{R}$,
3. $\theta(e) > 0$, for $e \in \hat{T}_0$,
4. $\eta \in P_a$ if and only if, for some $c > 0$ and $e \in \hat{T}_0$, η is of the form

$$\eta(z) = c \frac{\theta(z - a - e)\theta(z - \bar{a} + e)}{E(z, a)E(z, \bar{a})}.$$

Proof. See Fay [1, Chapter 6]. Here we note that, although we have chosen a different canonical homology basis, his results remain valid in our context. \square

Remark 2.1. If $e \in T_+$ and $e' \in T_+$ both represent the same η as in Lemma 2.1 (4), then from Riemann's vanishing theorem we see easily that $e = e' \pmod{2\pi i\mathbb{Z}^g}$. Here T_+ denotes a subset of \hat{T}_0 defined in Definition 3.1.

Lemma 2.2. $|\frac{E(x, y)}{E(x, \bar{y})}| = \exp(-g(x, y))$, for all $x, y \in R$.

Proof. First, we must clarify the meaning of the value of the left hand side of the lemma. By the multi-valuedness of $E(x, y)$, for fixed y , the value of the function $h(x) = |\frac{E(x, y)}{E(x, \bar{y})}|$ is meaningless unless we pose some restriction on the paths of integration for the divisors $y - x$ and $\bar{y} - x$. Thus we shall suppose that we always calculate the divisor $\bar{y} - x$ by decomposing $\bar{y} - x = (\bar{y} - y) + (y - x)$. Then by our

assumption stated above Lemma 2.1 we have $\bar{y} - y \in \hat{T}_0$. By using identities (2) and (3) we see that the function $h(x)$ is well-defined and single-valued on R . By Lemma 2.1 (2) $h(x) = 1$ on ∂R . Hence both the function $h(x) \exp(g(x, y))$ and its reciprocal are non-vanishing continuous subharmonic functions on R which are 1 on ∂R , so that $h(x) \exp(g(x, y)) \equiv 1$ on R as desired. \square

The following lemma except for equality condition was proved implicitly by H. Widom in [5, Theorem 5.6]. He applied it to the problems on the range of a functional associated to analytic functions with single-valued absolute value.

Lemma 2.3 (Widom). *Let $\{z_j\}_{j=1}^g$ be the zeroes of a positive differential in P_a . Then the inequality*

$$\sum_{j=1}^g g(z_j, a) \leq \sum_{j=1}^g g(z_j^*, a)$$

holds. The equality holds if and only if $\{z_j\}_{j=1}^g = \{z_j^\}_{j=1}^g$.*

Proof. Let $\{z_j\}_{j=1}^g$ be the zeroes of a differential η in P_a . By the residue theorem we may assume that η has a simple pole at a with residue $-i$. Since the functions $f(z) = \exp(-\sum_{j=1}^g g(z, z_j))$ and $f_0(z) = \exp(-\sum_{j=1}^g g(z, z_j^*))$ satisfy $f(z) = f_0(z) = 1$ on ∂R , we have

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_{\partial R} \eta = \frac{-1}{2\pi} \int_{\partial R} \frac{\eta}{i\omega_{\bar{a}-a}} \frac{\partial g}{\partial n} |dz| \\ &= \frac{-1}{2\pi} \int_{\partial R} \left| \frac{\eta}{i\omega_{\bar{a}-a}} \right| \frac{f_0}{f} \frac{\partial g}{\partial n} |dz|. \end{aligned}$$

Noting that the function $h(z) = \left| \frac{\eta(z)}{i\omega_{\bar{a}-a}(z)} \right| \frac{f_0(z)}{f(z)}$ is continuous and subharmonic on $R \cup \partial R$, the last integral is at least the value at a of the integrand, so that

$$1 \geq \lim_{z \rightarrow a} \left| \frac{\eta(z)}{i\omega_{\bar{a}-a}(z)} \right| \frac{f_0(z)}{f(z)} = \frac{f_0(a)}{f(a)},$$

which is the desired inequality. By maximum principle, equality holds if and only if the function $h(z)$ is harmonic. Since $h(z)$ is locally of the form $|k(z)|^2$ with some single-valued analytic function $k(z)$, we see that $h(z)$ is harmonic if and only if $h(z) \equiv \text{const}$ by the Cauchy-Riemann relation. Since $h(z) = \frac{\eta(z)}{i\omega_{\bar{a}-a}(z)}$ on ∂R , this implies that $\eta(z) = i\omega_{\bar{a}-a}(z)$ on R . Hence the equality statement is proved. \square

It is convenient to introduce the constants $e_0 = \sum_{j=1}^g z_j^* - a - \Delta \in \hat{T}_0$ and $e_1 = \sum_{j=1}^g \bar{z}_j^* - a - \Delta \in \hat{T}_0$ where Δ is the Riemann divisor class. For fixed a , these constants are determined modulo $2\pi i\mathbb{Z}^g$ and have the following extremal property.

Theorem 2.1. *For $a \in R$ and $e \in \hat{T}_0$, we have*

$$\exp\left(-\sum_{j=1}^g g(z_j^*, a)\right) \leq \theta(a - \bar{a} + e)/\theta(e) \leq \exp\left(\sum_{j=1}^g g(z_j^*, a)\right)$$

where $g(z, a)$ is the Green's function of R and the set $\{z_j^\}_{j=1}^g$ is the critical points of $g(\cdot, a)$. The equality occurs on the second (resp. first) inequality if and only if $e = e_0$ (resp. $e = e_1$) (mod $2\pi i\mathbb{Z}^g$).*

Proof. To begin with, we show that the first inequality is an easy consequence of the second. Since $\theta(z)$ is an even function, we have the symmetry

$$(5) \quad \theta(a - \bar{a} + e)/\theta(e) = \theta(e')/\theta(a - \bar{a} + e')$$

between $e \in \hat{T}_0$ and $e' = \bar{a} - a - e \in \hat{T}_0$.

On the other hand, it is classical that if $\sum d_j$ is the divisor of a meromorphic differential on a compact Riemann surface C , then $\sum d_j - 2\Delta = 0 \in J(C)$, the Jacobian variety of C [1, p. 7]. Applying this to the divisor of the differential $\omega_{\bar{a}-a}$ we obtain $\sum z_j^* + \sum \bar{z}_j^* - a - \bar{a} = 2\Delta$. Hence if $e = \sum_{j=1}^g z_j^* - a - \Delta$, then $e' = \sum_{j=1}^g \bar{z}_j^* - a - \Delta$. This suffices to conclude that we need only to prove the second inequality.

By the Jacobi Inversion Theorem, any $e \in \hat{T}_0$ can be written as

$$e = \sum_{j \in J} z_j + \sum_{j \in J'} \bar{z}_j - a - \Delta \in J(\hat{R})$$

where J is a subset of $\{1, 2, \dots, g\}$, $J' = \{1, 2, \dots, g\} \setminus J$ and $z_j \in R \cup \partial R$ for all $j = 1, 2, \dots, g$. Then from Riemann's vanishing theorem we obtain an identity

$$\frac{\theta(z - a - e)}{\theta(z - \bar{a} + e)} = \epsilon \prod_{j \in J} \frac{E(z, z_j)}{E(z, \bar{z}_j)} \prod_{j \in J'} \frac{E(z, \bar{z}_j)}{E(z, z_j)}$$

for some constant ϵ with absolute value 1. This is proved by observing that both sides of the above identity have single-valued absolute value with the same divisor and that they take absolute value 1 on ∂R . From Lemma 2.2 we have

$$\left| \frac{\theta(z - a - e)}{\theta(z - \bar{a} + e)} \right| = \exp \left(- \sum_{j \in J} g(z, z_j) + \sum_{j \in J'} g(z, z_j) \right).$$

Thus

$$(6) \quad \begin{aligned} \frac{\theta(e)}{\theta(a - \bar{a} + e)} &= \exp \left(- \sum_{j \in J} g(a, z_j) + \sum_{j \in J'} g(a, z_j) \right) \\ &\geq \exp \left(- \sum_{j=1}^g g(a, z_j) \right) = \frac{\theta(e^*)}{\theta(a - \bar{a} + e^*)}, \end{aligned}$$

where $e^* = \sum_{j=1}^g z_j - a - \Delta = e + \sum_{j \in J'} (z_j - \bar{z}_j) \in \hat{T}_0$. By Widom's Lemma, we have

$$(7) \quad \frac{\theta(e^*)}{\theta(a - \bar{a} + e^*)} \geq \frac{\theta(e_0)}{\theta(a - \bar{a} + e_0)}.$$

Combining the inequalities (6) and (7) we obtain the desired inequality. The equality statement is clear from Widom's Lemma. \square

3. HARDY H^2 AND CONJUGATE HARDY H^2 KERNELS

Definition 3.1. For fixed $a \in R$ let T_+ , T_{++} , T_- and T_{--} be the subsets of \hat{T}_0 given by

$$\begin{aligned} T_+ &= \mathcal{P} - a - \Delta \text{ and } T_{++} = \mathcal{P}_0 - a - \Delta, \\ T_- &= \overline{\mathcal{P}} - a - \Delta \text{ and } T_{--} = \overline{\mathcal{P}_0} - a - \Delta, \end{aligned}$$

where \mathcal{P} (resp. \mathcal{P}_0) is the set of zero divisors δ of degree g in $R \cup \partial R$ of a positive (resp. strictly positive) differential ω in P_a such that $\delta + \bar{\delta} - a - \bar{a} = \text{div}(\omega)$ on \hat{R} .

Since the values of the divisors on the right hand side of the above definition are determined only up to $2\pi i\mathbb{Z}^g$, we shall assume that T_+ , T_{++} , T_- and T_{--} are all $2\pi i\mathbb{Z}^g$ invariant. From Lemma 2.1 (4) it is easy to show that T_{++} is an open subset of \hat{T}_0 and that $\text{Cl}(T_{++}) = T_+$. A similar assertion holds for the sets T_- and T_{--} .

Example 3.1 (Case $g = 1$). Let R be a region obtained by identifying the vertical sides of a rectangle Q with vertices at $0, \tau, \tau - \pi i, -\pi i$ ($\tau < 0$). Then R is doubly connected and its Schottky double \hat{R} is a torus obtained by identifying the sides of a rectangle $Q \cup \bar{Q}$ as usual where \bar{Q} denotes the reflection of Q in the real axis. Also dz is a normalized differential of the first kind on \hat{R} with period matrix $(2\pi i, \tau)$. It follows from the fact $\Delta = \frac{\tau}{2} + \pi i$ ([1, p. 14]) and Lemma 2.1 (4) that, for $a = -ci \in R$ ($0 < c < \pi$), the zero of a differential in P_a has real part $\tau/2$. Setting $e = z - a - \Delta \in \hat{T}_0$ with $z = \tau/2 - id \in R$ ($0 < d < \pi$) we see easily that

$$T_+ \setminus T_{++} = T_- \setminus T_{--} = \{(n\pi + c)i \mid n \in \mathbb{Z}\}$$

and

$$T_{++} = \bigcup_{n \in \mathbb{Z}} iI_{2n}, \quad T_{--} = \bigcup_{n \in \mathbb{Z}} iI_{2n+1} \quad (\subset i\mathbb{R})$$

with the interval $I_n = (n\pi + c, (n+1)\pi + c)$.

We now define two weighted Hardy spaces on R parametrized by the set T_{++} .

Definition 3.2. For $e \in T_{++}$ let

$$(8) \quad \omega_e(x) = \frac{i\theta(x-a-e)\theta(x-\bar{a}+e)E(a,\bar{a})}{\theta(e)\theta(a-\bar{a}+e)E(x,a)E(x,\bar{a})}$$

be a strictly positive differential on R with simple poles of residue $-i, i$ at a, \bar{a} . Denote by $H_e^2(R)$ the Hilbert space of holomorphic functions $f(z)$ on R such that the function $|f(z)|^2$ admits a harmonic majorant on R . For $f, g \in H_e^2(R)$ the inner product is defined by $\langle f, g \rangle_e = \frac{1}{2\pi} \int_{\partial R} f \bar{g} \omega_e$. Also, denote by $H_e^{2,1}(R)$ the Hilbert space of holomorphic differentials $\xi(z)$ on R such that $\xi(z)/dz \in H_e^2(R)$. For $\xi, \eta \in H_e^{2,1}(R)$ the inner product is defined by $\langle \xi, \eta \rangle_{e,1} = \frac{1}{2\pi} \int_{\partial R} \xi \bar{\eta} / \omega_e$.

Lemma 3.1. The Hilbert spaces $H_e^2(R)$ and $H_e^{2,1}(R)$ possess the reproducing kernels $R_e(x, \bar{y})$ and $\hat{R}_e(x, \bar{y})$ respectively given by:

$$(9) \quad R_e(x, \bar{y}) = \frac{\theta(a-\bar{a}+e)\theta(x-\bar{y}+e)}{\theta(x-\bar{a}+e)\theta(a-\bar{y}+e)} \frac{E(x, \bar{a})E(\bar{y}, a)}{E(\bar{a}, a)E(x, \bar{y})}, \quad x, y \in R,$$

$$(10) \quad \hat{R}_e(x, \bar{y}) = \frac{\theta(x-\bar{a}+e)\theta(a-\bar{y}+e)\theta(x-\bar{y}-e)E(a, \bar{a})}{\theta^2(e)\theta(a-\bar{a}+e)E(x, \bar{y})E(x, \bar{a})E(\bar{y}, a)}, \quad x, y \in R.$$

In particular,

$$(11) \quad \hat{R}_e(x, \bar{y}) = -R_e(\bar{y}, x)\omega_e(x)\omega_e(\bar{y}).$$

Proof. From the residue theorem a simple computation shows that both (9) and (10) have the reproducing property of the corresponding Hilbert spaces (cf. Fay [Proposition 6.15]). The details may be left to the reader. \square

We shall call $R_e(x, \bar{y})$ the *Hardy H^2 kernel* and $\hat{R}_e(x, \bar{y})$ the *conjugate Hardy H^2 kernel* associated with $e \in T_{++}$ (cf. [4, Section 3 of chapter III]).

Remark 3.1. If $e = e_0$, then $R_{e_0}(x, \bar{y})$ and $\hat{R}_{e_0}(x, \bar{y})$ are the usual Hardy H^2 kernel and its conjugate kernel. In this special case the results in the present section were partly obtained in the author's report [6].

Let $H_{e,a}^2(R)$ be the subspace $\{f \in H_e^2(R) | f(a) = 0\}$ of $H_e^2(R)$. For $f \in H_{e,a}^2(R)$ define the mapping $l: H_{e,a}^2(R) \rightarrow H_e^{2,1}(R)$ by $l(f) = f\omega_e$. It follows at once that the mapping l is a complex linear isometry from $H_{e,a}^2(R)$ into $H_e^{2,1}(R)$. Identifying the spaces $H_{e,a}^2(R)$ and $l(H_{e,a}^2(R))$ via l we may regard $H_{e,a}^2(R)$ as a subspace of $H_e^{2,1}(R)$.

Lemma 3.2. *The following orthogonal decomposition holds:*

$$(12) \quad H_e^{2,1}(R) = H_{e,a}^2(R) \oplus \Gamma(\hat{R})$$

where $\Gamma(\hat{R})$ denotes the space of holomorphic differentials on R which extend to Abelian differentials of the first kind on \hat{R} .

Proof. Assume that $\xi \in H_e^{2,1}(R)$ is orthogonal to $H_{e,a}^2(R)$. Then, for any $f \in H_{e,a}^2(R)$ we have

$$0 = \langle (z-a)f\omega_e, \xi \rangle_{e,1} = \frac{1}{2\pi} \int_{\partial R} (z-a)f(z)\overline{\xi(z)},$$

since $(z-a)f(z) \in H_{e,a}^2(R)$. By the Cauchy-Riemann Theorem [3], there exists a holomorphic differential $\eta \in H_e^{2,1}(R)$ such that $(z-a)\overline{\xi(z)} = \eta(z)$, $z \in \partial R$. This means that ξ is extended to an Abelian differential on \hat{R} with at most a simple pole at \bar{a} and elsewhere regular. Since it is well-known that the sum of the residues of an Abelian differential equals to zero, ξ belongs to $\Gamma(\hat{R})$, as desired.

The converse is proved at once by applying the Cauchy integral theorem. \square

Since $R_e(a, \bar{y}) \equiv 1$, the reproducing kernel of $H_{e,a}^2(R)$ as a subspace of $H_e^2(R)$ is given by $R_e(x, \bar{y}) - 1$. Thus, by means of the orthogonal decomposition (12) we have

$$(13) \quad \hat{R}_e(x, \bar{y}) = (R_e(x, \bar{y}) - 1)\omega_e(x)\omega_e(\bar{y}) + \sum_{i,j=1}^g c_{ij}u_i(x)u_j(\bar{y})$$

with some positive definite matrix $(c_{ij})_{i,j=1}^g$.

For the sake of simplicity, we introduce the notation: for $x, y \in \hat{R}$ and $e \in \mathbb{C}^g$

$$(x, y)_e = \frac{\theta(x - y - e)}{\theta(e)E(x, y)}.$$

When $e = 0$, the expression $\frac{1}{2\pi i}(x, \bar{y})_0$ is known to coincide with the classical Szegő kernel $\hat{K}(x, \bar{y})$ [1], [2]. If the subscript ' e ' is clear from the context, we shall omit

it. In view of this notation, formulas (8), (9) and (10) are given by

$$(14) \quad \omega_e(x) = i \frac{(\bar{a}, x)(x, a)}{(\bar{a}, a)},$$

$$(15) \quad R_e(x, \bar{y}) = \frac{(\bar{a}, a)(\bar{y}, x)}{(\bar{a}, x)(\bar{y}, a)},$$

$$(16) \quad \hat{R}_e(x, \bar{y}) = \frac{(\bar{a}, x)(x, \bar{y})(\bar{y}, a)}{(\bar{a}, a)}.$$

Substituting formulas (14)-(16) to (13) we obtain

$$(17) \quad \frac{(\bar{a}, x)(x, \bar{y})(\bar{y}, a)}{(\bar{a}, a)} + \frac{(\bar{a}, \bar{y})(\bar{y}, x)(x, a)}{(\bar{a}, a)} - \frac{(\bar{a}, x)(x, a)(\bar{a}, \bar{y})(\bar{y}, a)}{(\bar{a}, a)^2} = \sum_{i,j=1}^g c_{ij} u_i(x) u_j(\bar{y}).$$

To determine the matrix (c_{ij}) explicitly, we need the following addition-theorems.

Lemma 3.3 (Fay). *For $a, b, x, y \in \hat{R}$ and $e \in \mathbb{C}^g$ with $\theta(e) \neq 0$, we have*

$$(18) \quad (x, a)(y, b) - (x, b)(y, a) = \frac{\theta(x+y-a-b-e)E(x, y)E(b, a)}{\theta(e)E(x, a)E(x, b)E(y, a)E(y, b)},$$

$$(19) \quad \frac{(a, x)(x, b)}{(a, b)} = \omega_{a-b}(x) - \sum_{i=1}^g \left\{ \frac{\partial \log \theta}{\partial z_i}(b-a+e) - \frac{\partial \log \theta}{\partial z_i}(e) \right\} u_i(x).$$

Proof. (18) is Fay's trisecant formula [1, formula (45) in p.34]. (19) is nothing but Proposition 2.10 of his book [1, formula (38) in p.25]. \square

Remark 3.2. As Fay pointed out in his book, formula (19) is a specialization of (18). In fact, dividing both sides of (18) by (x, a) and then letting $y \rightarrow b$, we obtain (19) in view of (4) and L'Hospital's rule.

Theorem 3.1. *For $a, b, x, y \in \hat{R}$ and $e \in \mathbb{C}^g$ with $\theta(e) \neq 0$, we have*

$$(20) \quad \frac{(a, x)(x, y)(y, b)}{(a, b)} + \frac{(a, y)(y, x)(x, b)}{(a, b)} - \frac{(a, x)(x, b)(a, y)(y, b)}{(a, b)^2} \\ = \sum_{i,j=1}^g \left\{ \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(a-b-e) - \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) \right\} u_i(x) u_j(y).$$

Proof. Let I be the left hand side of (20) and write $I = I_1 + I_2 - I_3$ in the obvious manner. Then by (18)

$$I_2 - I_3 = \frac{(a, y)(x, b)}{(a, b)^2} ((a, b)(y, x) - (a, x)(y, b)) \\ = \frac{\theta(a-y-e)\theta(x-b-e)\theta(y-x+a-b-e)E(a, b)}{\theta(e)\theta^2(a-b-e)E(a, x)E(y, x)E(y, b)} \\ = -\frac{(a, x)_f(x, y)_f(y, b)_f}{(a, b)_f},$$

where we put $f = a - b - e \in \mathbb{C}^g$. Using the trivial identity

$$\frac{(a, x)_e(x, b)_e}{(a, b)_e} = \frac{(a, x)_f(x, b)_f}{(a, b)_f}$$

we have

$$\begin{aligned}
 I &= \frac{(a, x)_e (x, b)_e}{(a, b)_e} \left\{ \frac{(x, y)_e (y, b)_e}{(x, b)_e} - \frac{(x, y)_f (y, b)_f}{(x, b)_f} \right\} \\
 &= -\frac{(a, x)_e (x, b)_e}{(a, b)_e} \sum_{j=1}^g \left\{ \frac{\partial}{\partial z_j} \log \frac{\theta(z+b-x)}{\theta(z)} \Big|_e - \frac{\partial}{\partial z_j} \log \frac{\theta(z+b-x)}{\theta(z)} \Big|_f \right\} u_j(y) \\
 &= -\frac{(a, x)_e (x, b)_e}{(a, b)_e} \sum_{j=1}^g \frac{\partial}{\partial z_j} \log \frac{\theta(z+b-x)\theta(-z+a-x)}{\theta(z)\theta(a-b-z)} \Big|_e u_j(y) \\
 &= -\frac{(a, x)_e (x, b)_e}{(a, b)_e} \sum_{j=1}^g \frac{\partial}{\partial z_j} \log \frac{(a, x)_z (x, b)_z}{(a, b)_z} \Big|_e u_j(y) \\
 &= -\sum_{j=1}^g \frac{\partial}{\partial z_j} \frac{(a, x)_z (x, b)_z}{(a, b)_z} \Big|_e u_j(y) = \sum_{i,j=1}^g \frac{\partial^2}{\partial z_i \partial z_j} \log \frac{\theta(z+b-a)}{\theta(z)} \Big|_e u_i(x) u_j(y).
 \end{aligned}$$

Note that in the second and the last equality we have applied the formula (19). This completes the proof. \square

Corollary 3.1. *The matrix (c_{ij}) in (13) is given by*

$$c_{ij} = \frac{\partial^2}{\partial z_i \partial z_j} \log \frac{\theta(a - \bar{a} + z)}{\theta(z)} \Big|_e, \quad i, j = 1, \dots, g.$$

Corollary 3.2. *For $a, b, c \in \hat{R}$ and $e \in \mathbb{C}^g$ with $\theta(e) \neq 0$, we have*

$$(a, b)(b, c)(c, a) + (a, c)(c, b)(b, a) = \sum_{i,j,k=1}^g \frac{\partial^3 \log \theta}{\partial z_i \partial z_j \partial z_k}(e) u_i(a) u_j(b) u_k(c).$$

Proof. Letting $a \rightarrow b$ after multiplying (a, b) to both sides of (20), we have

$$\begin{aligned}
 &(b, x)(x, y)(y, b) + (b, y)(y, x)(x, b) \\
 &= \lim_{a \rightarrow b} \sum_{i,j=1}^g \left\{ \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(a - b - e) - \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) \right\} \frac{u_i(x) u_j(y)}{E(a, b)}.
 \end{aligned}$$

Noting that $\lim_{a \rightarrow b} \frac{E(a, b)}{a - b} = -1$ (cf. [1, p.19 (26)]) and the function $\frac{\partial^3 \log \theta}{\partial z_i \partial z_j \partial z_k}(z)$ is odd, we obtain the corollary at once from L'Hospital's rule. \square

Theorem 3.2. *For $e \in \hat{T}_0$ and $a \in R$ let Φ be the $g \times g$ matrix*

$$\left(\frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(a - \bar{a} + e) - \frac{\partial^2 \log \theta}{\partial z_i \partial z_j}(e) \right).$$

Then, Φ is a real matrix and the following holds.

1. *For any $e \in T_+$ the matrix Φ is positive semi-definite. Φ is positive definite for $e \in T_+$ if and only if $e \in T_{++}$.*
2. *For any $e \in T_-$ the matrix Φ is negative semi-definite. Φ is negative definite for $e \in T_-$ if and only if $e \in T_{--}$.*

Proof. It is clear from Lemma 2.1 (3) and analyticity of $\theta(z)$ that the matrix Φ is real for any $e \in \hat{T}_0$. By symmetry (5) item 2 follows at once from item 1.

Now we shall prove item 1. If $e \in T_{++}$, then (13) and Corollary 3.1 show that Φ is positive definite. Since $\text{Cl}(T_{++}) = T_+$, this suffices to prove the first statement

and the only if part of the second statement. To complete the proof we need only to show that if $e \in T_+ \setminus T_{++}$, then Φ is singular. Hence suppose that $e \in T_+ \setminus T_{++}$ and define the differential ω_e as in (8). By definition ω_e is a positive differential with a zero w on ∂R . This implies $(w, a) = 0$. Then we have also $(\bar{a}, w) = 0$, since for all $x, y \in \hat{R}$ the identity $\overline{(x, y)} = -(\bar{y}, \bar{x})$ holds. From (17) we obtain

$$\sum_{i,j=1}^g c_{ij} u_i(x) u_j(w) = 0$$

for all $x \in \hat{R}$. Since the differentials $\{u_i(x)\}_{i=1}^g$ are linearly independent, this implies

$$\sum_{j=1}^g c_{ij} u_j(w) = 0$$

for all $i = 1, \dots, g$. But it is classical that $u_j(w) \neq 0$ for some $j = 1, \dots, g$. Thus Φ is singular, as desired. \square

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