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DEGENERATIONS FOR MODULES OVER REPRESENTATION-FINITE ALGEBRAS

GRZEGORZ ZWARA

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Abstract. Let A be a representation-finite algebra. We show that a finite dimensional A-module M degenerates to another A-module N if and only if the inequalities $\dim_K Hom_A(M,X) \leq \dim_K Hom_A(N,X)$ hold for all A-modules X. We prove also that if $\mathrm{Ext}_A^1(X,X)=0$ for any indecomposable A-module X, then any degeneration of A-modules is given by a chain of short exact sequences.

1. Introduction and main results

Let A be a finite dimensional associative K-algebra with an identity over an algebraically closed field K. If $a_1 = 1, \dots, a_n$ is a basis of A over K, we have the structure constants a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\mathrm{mod}_A(d)$ of d-dimensional unital left A-modules consists of n-tuples $m = (m_1, \dots, m_n)$ of $d \times d$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices i and j. The general linear group $\mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [11]). We shall agree to identify a d-dimensional A-module M with the point of $\mathrm{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\mathrm{Gl}_d(K)$-orbit of a module M in $\mathrm{mod}_A(d)$. Then one says that a module N in $\mathrm{mod}_A(d)$ is a degeneration of a module M in $\mathrm{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(d)$, and we denote this fact by $M \leq_{\mathrm{deg}} N$. Thus $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\leq_{\mathrm{deg}}$ in terms of representation theory.

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [6], [9], [8], Ch. Riedtmann [13], and A. Skowroński and the author [15], [16], [17] and [18] connecting $\leq_{\mathrm{deg}}$ with other partial orders $\leq_{\mathrm{ext}}$ and $\leq$ on the isomorphism classes in $\mathrm{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \leq_{\mathrm{ext}} N$: $\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\mathrm{mod}\, A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq N$: $\Leftrightarrow$ $[M,X] \leq [N,X]$ holds for all modules X.

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Here and later on we abbreviate $\dim_K \operatorname{Hom}_A(X,Y)$ by $[X,Y]$. Then for modules M and N in $\operatorname{mod}_A(d)$, the following implications hold:

$$M \leq_{\text{ext}} N \Longrightarrow M \leq_{\text{deg}} N \Longrightarrow M \leq N$$

(see [9], [13]). Unfortunately, the reverse implications are not true in general, and it would be interesting to find out when they are. K. Bongartz proved in [9] that it is the case for all representations of Dynkin quivers and the double arrow. Moreover, in [8] K. Bongartz proved that $\leq_{\text{deg}}$ and $\leq$ coincide for all modules over tame concealed algebras. Recently, the author proved in [17] that $\leq$ and $\leq_{\text{ext}}$ are also equivalent for all modules over representation-finite blocks of group algebras, and in [18] that $\leq_{\text{ext}}$ and $\leq_{\text{deg}}$ coincide for all modules over tame concealed algebras. The main aim of this paper is to prove the following theorem.

Theorem 1. *Let A be a representation-finite algebra and M, N two modules with $M \leq N$. Then there are A-modules Z, Z', and two exact sequences*

$$0 \to N \to M \oplus Z \to Z \to 0 \quad \textit{and} \quad 0 \to Z' \to M \oplus Z' \to N \to 0.$$

In [13] Riedtmann proved that each of the exact sequences $0 \to N \to M \oplus Z \to Z \to 0$ and $0 \to Z' \to M \oplus Z' \to N \to 0$ implies that $M \leq_{\text{deg}} N$. Hence we get the following fact which solves a long standing problem (see [13]).

Corollary. *The partial orders $\leq$ and $\leq_{\text{deg}}$ coincide for all modules over representation-finite algebras.*

We note that for a representation-finite algebra A we may deduce the dimension of the spaces $Hom_A(M,N)$ from the Auslander-Reiten quiver of A (see [10]), and hence it is rather easy to decide when $M \leq N$ for any A-modules M and N.

There are many examples of representation-finite algebras for which the orders $\leq_{\text{deg}}$ and $\leq_{\text{ext}}$ are not equivalent (see [17]). Our second aim in this paper is to prove the following theorem.

Theorem 2. *Let B be an algebra and assume that $\operatorname{Ext}^1_B(X,X) = 0$ for any indecomposable B-module X. Then the partial orders $\leq$, $\leq_{\text{deg}}$ and $\leq_{\text{ext}}$ coincide for all B-modules.*

It is well-known that every representation-directed algebra [14] satisfies the above condition. Hence, Theorem 2 extends the corresponding result by Bongartz proved in [9].

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. Section 3 is devoted to the proofs of Theorems 1 and 2.

For basic background on the topics considered here we refer to [5], [9], [8], [11] and [14]. The results presented in this paper form a part of the author's doctoral dissertation written under the supervision of Professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

2. Preliminary results

2.1. Throughout the paper A denotes a fixed finite dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite dimensional left A-modules and by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$. By an A-module we mean an object from $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander-Reiten quiver of A, and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander-Reiten translations $D \operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We shall agree to identify the vertices of Γ_A with the corresponding indecomposable modules. For a module M we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if M and N have the same simple composition factors including the multiplicities.

2.2. Following [13], for M, N from $\operatorname{mod} A$, we set $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all A-modules X. The fact that $\leq$ is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander [3] (see also [6]). Observe that, if M and N have the same dimension and $M \leq N$, then $[M] = [N]$. Moreover, M. Auslander and I. Reiten have shown in [4] that, if M and N are A-modules with $[M] = [N]$, then for all nonprojective indecomposable A-modules X and all noninjective indecomposable modules Y the following formulas hold (see [12]):

$$[X, M] - [M, \tau X] = [X, N] - [N, \tau X],$$
$$[M, Y] - [\tau^- Y, M] = [N, Y] - [\tau^- Y, N].$$

Hence, if $[M] = [N]$, then $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all A-modules X.

2.3. Let M and N be A-modules with $[M] = [N]$ and

$$\Sigma : \ 0 \to D \to E \to F \to 0$$

an exact sequence in $\operatorname{mod} A$. Following [13] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$ and δ_Σ on A-modules X as follows:

$$\begin{aligned}
\delta_{M,N}(X) &= [N, X] - [M, X],\\
\delta'_{M,N}(X) &= [X, N] - [X, M],\\
\delta_\Sigma(X) &= \delta_{E, D\oplus F}(X) = [D \oplus F, X] - [E, X],\\
\delta'_\Sigma(X) &= \delta'_{E, D\oplus F}(X) = [X, D \oplus F] - [X, E].
\end{aligned}$$

From the Auslander-Reiten formulas (2.2) we get the following very useful equalities:

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \qquad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, the following conditions are equivalent:

(1) $M \leq N$,
(2) $\delta_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$,
(3) $\delta'_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$.

2.4. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M, Z)$ the multiplicity of Z as a direct summand of M. For a noninjective indecomposable A-module U, we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U) : \ 0 \to U \to E(U) \to \tau^- U \to 0.$$

We shall need the following lemma.

Lemma 2.5. *Let M, N be A-modules with $[M] = [N]$ and U an indecomposable A-module. Then:*

(i) *If U is noninjective, then $\delta_{\Sigma(U)}(M) = \mu(M, U)$.*
(ii) *If $M \leq N$, then $\mu(N, U) - \mu(M, U) \leq \delta_{M,N}(U) + \delta'_{M,N}(U)$.*

Proof. If U is noninjective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \mathrm{Hom}_A(\tau^- U, M) \to \mathrm{Hom}_A(E(U), M) \to \mathrm{rad}(U, M) \to 0,$$

and hence we get

$$[U \oplus \tau^- U, M] - [E(U), M] = [U, M] - \dim_K \mathrm{rad}(U, M) = \mu(M, U).$$

This implies (i). Similarly, we have

$$[U \oplus \tau^- U, N] - [E(U), N] = \mu(N, U).$$

Then we obtain

$$\begin{aligned} \mu(N, U) - \mu(M, U) &= ([U \oplus \tau^- U, N] - [U \oplus \tau^- U, M]) - ([E(U), N] - [E(U), M]) \\ &= \delta'_{M,N}(U) + \delta'_{M,N}(\tau^- U) - \delta'_{M,N}(E(U)) \\ &\leq \delta'_{M,N}(U) + \delta'_{M,N}(\tau^- U) = \delta'_{M,N}(U) + \delta_{M,N}(U). \end{aligned}$$

Assume now that U is injective. Then $\mathrm{Hom}_A(U/\mathrm{soc}(U), M) \simeq \mathrm{rad}(U, M)$, and so

$$[U, M] - [U/\mathrm{soc}(U), M] = \mu(M, U).$$

Similarly, we have

$$[U, N] - [U/\mathrm{soc}(U), N] = \mu(N, U).$$

Therefore, we get

$$\begin{aligned} \mu(N, U) - \mu(M, U) &= ([U, N] - [U, M]) - ([U/\mathrm{soc}(U), N] - [U/\mathrm{soc}(U), M]) \\ &= \delta'_{M,N}(U) - \delta'_{M,N}(U/\mathrm{soc}(U)) \leq \delta'_{M,N}(U) \\ &= \delta'_{M,N}(U) + \delta_{M,N}(U). \end{aligned}$$

Hence, (ii) also holds. □

We shall need also the following Lemma $(3+3+2)$ from [2, (2.1)] and its direct consequence.

Lemma 2.6. *Let*

$$\Sigma_1 : \ 0 \to M_1 \xrightarrow{\begin{bmatrix} u_1 \\ f_1 \end{bmatrix}} M_2 \oplus N_1 \xrightarrow{[f_2, u_2]} N_2 \to 0,$$
$$\Sigma_2 : \ 0 \to M_2 \xrightarrow{\begin{bmatrix} v_1 \\ f_2 \end{bmatrix}} M_3 \oplus N_2 \xrightarrow{[f_3, v_2]} N_3 \to 0$$

be short exact sequences in mod A. *Then the sequence*

$$\Sigma_3 : \ 0 \to M_1 \xrightarrow{\begin{bmatrix} v_1 u_1 \\ f_1 \end{bmatrix}} M_3 \oplus N_1 \xrightarrow{[f_3, -v_2 u_2]} N_3 \to 0$$

is exact. Moreover, we have $\delta_{\Sigma_3} = \delta_{\Sigma_1} + \delta_{\Sigma_2}$.

2.7. A short exact sequence

$$0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$$

is said to be *without isomorphism* provided $f \in \operatorname{rad}(U,W)$ and $g \in \operatorname{rad}(W,V)$. Let $\Sigma : 0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$ be any exact sequence. It is easy to see that if $f \in \operatorname{rad}(U,W)$, then there is an exact sequence without isomorphism $0 \to U \to W' \to V' \to 0$ such that $W = W' \oplus Y$ and $V = V' \oplus Y$ for some A-modules W', V' and Y. Dually, if $g \in \operatorname{rad}(W,V)$, then there is an exact sequence without isomorphism $0 \to U' \to W' \to V \to 0$ such that $U = U' \oplus Z$ and $W = W' \oplus Z$ for some A-modules U', W' and Z. Moreover, if Σ is nonsplittable, then there is a nonsplittable exact sequence without isomorphism $0 \to U' \to W' \to V' \to 0$ such that $U = U' \oplus Y$, $W = W' \oplus Y \oplus Z$ and $V = V' \oplus Z$ for some A-modules U', W', V', Y and Z.

Lemma 2.8. *Let $\Sigma : 0 \to U \xrightarrow{f} W \xrightarrow{g} V \to 0$ be an exact sequence without isomorphism. Then:*

(i) *For any nonzero direct summand U' of U, $\delta_\Sigma(U') > 0$ holds.*
(ii) *For any nonzero direct summand V' of V, $\delta'_\Sigma(V') > 0$ holds.*

Proof. (i) Let U' be a nonzero direct summand of U. The sequence Σ induces an exact sequence

$$0 \to \operatorname{Hom}_A(V,U') \to \operatorname{Hom}_A(W,U') \xrightarrow{f^*} \operatorname{Hom}_A(U,U').$$

Assume that f^* is an epimorphism. Then there is a homomorphism of A-modules $h : W \to U'$ such that $f^*(h) = hf : U \to U'$ is a projection. But then $f \notin \operatorname{rad}(U,W)$, which yields a contradiction. Hence, $[V,U'] - [W,U'] + [U,U'] > 0$, and consequently $\delta_\Sigma(U') > 0$.

The proof of (ii) is dual. $\square$

As a consequence of the above lemma, we get the following fact.

Lemma 2.9. *Let $\Sigma : 0 \to U \to W \to V \to 0$ be a nonsplittable exact sequence. Then $\delta_\Sigma(U) > 0$ and $\delta'_\Sigma(V) > 0$.*

Lemma 2.10. *Let X be an A-module and $\Sigma : 0 \to U \to W \to V \to 0$ a nonsplittable short exact sequence of A-modules.*

(i) *If $\delta_\Sigma(X) > 0$, then there exists a nonsplittable exact sequence of A-modules*

$$\Phi : 0 \to X \to Y \to V \to 0,$$

such that $\delta_\Phi \le \delta_\Sigma$.

(ii) *If $\delta'_\Sigma(X) > 0$, then there exists a nonsplittable exact sequence of A-modules*

$$\Phi : 0 \to U \to Z \to X \to 0,$$

such that $\delta_\Phi \le \delta_\Sigma$.

Proof. (i) The first part of the proof is due to U. Markolf (see the proof of Theorem 4 in [7]). Let X be an A-module such that $\delta_\Sigma(X) > 0$. Then the last map in the following exact sequence

$$0 \to \operatorname{Hom}_A(V,X) \to \operatorname{Hom}_A(W,X) \to \operatorname{Hom}_A(U,X) \to \operatorname{Ext}^1_A(V,X) \to \operatorname{Ext}^1_A(W,X)$$

is not a monomorphism. Therefore, we find a nonsplittable exact sequence of A-modules $\Phi : 0 \to X \to Y \to V \to 0$, whose pullback under $W \to V$ is a splittable sequence. Thus we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & U & = & U & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & X & \longrightarrow & X \oplus W & \longrightarrow & W & \to 0 \\ & \| & & \downarrow & & \downarrow & \\ 0 \to & X & \longrightarrow & Y & \longrightarrow & V & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

So, we have an exact sequence $\Theta : 0 \to U \to X \oplus W \to Y \to 0$. Observe that $\delta_\Sigma = \delta_\Phi + \delta_\Theta$. This implies that $\delta_\Phi \leq \delta_\Sigma$.

The proof of (ii) is dual. □

Lemma 2.11. *If $M <_{\deg} N$, then $\delta_{M,N}(N) > 0$ and $\delta'_{M,N}(N) > 0$.*

Proof. Suppose that $\delta'_{M,N}(N) = 0$. By Theorem 2.4 in [9], we know that if a module U embeds into N and $[U, N] = [U, M]$, then U also embeds into M. Applying this fact for $U = N$, we obtain that N embeds into M. But the modules M and N have the same dimension. This implies that M is isomorphic to N, which gives a contradiction. Hence, $\delta'_{M,N}(N) > 0$ and $\delta_{M,N}(N) > 0$ by duality. □

3. Proof of Theorems 1 and 2

Throughout this section A denotes a representation-finite algebra.

Lemma 3.1. *Let M and N be two A-modules with $M < N$, and let*

$$\Sigma : 0 \to U \to W \to V \to 0$$

be a short exact sequence without isomorphism in $\operatorname{mod} A$ *such that $\delta_\Sigma \leq \delta_{M,N}$. Then there exists a short exact sequence without isomorphism in* $\operatorname{mod} A$

$$\Phi : 0 \to U \to Y \to Z \to 0$$

such that $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(Y) = \delta_{M,N}(Y)$.

Proof. Let

$$\Sigma : 0 \to U \to W \to V \to 0$$

be a short exact sequence of A-modules without isomorphism such that $\delta_\Sigma \leq \delta_{M,N}$. Take a short exact sequence without isomorphism in $\operatorname{mod} A$,

$$\Phi : 0 \to U \to Y \to Z \to 0$$

such that $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$, and which is maximal in the following sense. For any short exact sequence without isomorphism Φ' in $\operatorname{mod} A$ starting at U and satisfying inequalities $\delta_\Phi \leq \delta_{\Phi'} \leq \delta_{M,N}$, we have $\delta_\Phi = \delta_{\Phi'}$. Since $\sum_{X \in \Gamma_A} \delta_{M,N}(X)$ is finite, such a sequence Φ exists. Assume now that $Y = Y_1 \oplus Y_2$, where Y_1 is indecomposable with $\delta_\Phi(Y_1) < \delta_{M,N}(Y_1)$. Then Y_1 is noninjective and we have an Auslander-Reiten sequence

$$\Sigma(Y_1) : 0 \to Y_1 \xrightarrow{h} E \to \tau^- Y_1 \to 0,$$

and of course

$$\Phi : 0 \to U \to Y_1 \oplus Y_2 \xrightarrow{(f_1,f_2)} Z \to 0.$$

Since $f_1 \in \operatorname{rad}(Y_1, Z)$, the push out of the Auslander-Reiten sequence $\Sigma(Y_1)$ is a splittable sequence, so we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & Y_1 & \xrightarrow{h} & E & \longrightarrow & \tau^- Y_1 & \to 0 \\ & \downarrow f_1 & & \downarrow & & || & \\ 0 \to & Z & \longrightarrow & \tau^- Y_1 \oplus Z & \longrightarrow & \tau^- Y_1 & \to 0. \end{array}$$

This implies that there exists a nonsplittable exact sequence

$$\Psi : 0 \to Y_1 \xrightarrow{\begin{pmatrix} h \\ f_1 \end{pmatrix}} E \oplus Z \to \tau^- Y_1 \oplus Z \to 0.$$

Applying Lemma 2.6 for Φ and Ψ, we get a new exact sequence

$$0 \to U \xrightarrow{\imath} Y_2 \oplus E \to Z \oplus \tau^- Y_1 \to 0.$$

Since Φ is a sequence without isomorphism, we have $\imath \in \operatorname{rad}(U, Y_2 \oplus E)$. Hence, there is a sequence without isomorphism in $\operatorname{mod} A$

$$\Theta : 0 \to U \to \overline{Y} \to \overline{Z} \to 0,$$

with $Y_2 \oplus E = \overline{Y} \oplus \overline{W}$ and $Z \oplus \tau^- Y_1 = \overline{Z} \oplus \overline{W}$ for some A-module $\overline{W}$. Thus, by Lemmas 2.6 and 2.5(i), for any A-module X we have

$$\delta_\Theta(X) = \delta_\Phi(X) + \delta_\Psi(X) = \delta_\Phi(X) + \delta_{\Sigma(Y_1)}(X) = \delta_\Phi(X) + \mu(X, Y_1).$$

Since $\delta_\Phi \le \delta_{M,N}$ and $\delta_\Phi(Y_1) \le \delta_{M,N}(Y_1) - 1$, we get $\delta_\Sigma \le \delta_\Theta \le \delta_{M,N}$. This gives a contradiction with our choice of the sequence Φ. Hence, $\delta_\Phi(Y) = \delta_{M,N}(Y)$, and this finishes the proof. □

Lemma 3.2. *If $M < N$, then $\delta_{M,N}(N) > 0$ and $\delta'_{M,N}(N) > 0$.*

Proof. We proceed by induction on $\sum_{X \in \Gamma_A} \delta_{M,N}(X) > 0$. Applying equalities (2.3), we obtain $\sum_{X \in \Gamma_A} \delta_{M,N}(X) = \sum_{X \in \Gamma_A} \delta'_{M,N}(X)$. Assume $M < N$ and that $\delta_{M,N}(N) = 0$ or $\delta'_{M,N}(N) = 0$. By duality, we may assume that $\delta'_{M,N}(N) = 0$ and moreover, the modules M and N have no nonzero common direct summand. Let $\mathcal{F}$ be the set of all modules in Γ_A which are a direct summands of N. Take $Y \in \mathcal{F}$. By Lemma 2.5(ii), we get

$$\mu(N, Y) = \mu(N, Y) - \mu(M, Y) \le \delta_{M,N}(Y) + \delta'_{M,N}(Y) = \delta_{M,N}(Y).$$

So, the module Y is noninjective and there is an Auslander-Reiten sequence $\Sigma(Y)$. We define a new exact sequence without isomorphism

$$\Sigma : 0 \to N \to E(N) \to \tau^- N \to 0,$$

where $E(N) = \bigoplus_{Y \in \mathcal{F}} E(Y)^{\mu(N,Y)}$ and $\tau^- N = \bigoplus_{Y \in \mathcal{F}} (\tau^- Y)^{\mu(N,Y)}$. Applying Lemma 2.5(i), we obtain

$$\delta_\Sigma(Y) = \mu(N, Y) \le \delta_{M,N}(Y),$$

for any $Y \in \Gamma_A$. Consequently $\delta_\Sigma \le \delta_{M,N}$ and, from Lemma 3.1, there is an exact sequence without isomorphism

$$\Phi : 0 \to N \to W \to V \to 0$$

with $\delta_\Phi \le \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then $M \oplus V \le W$ and $\delta_{M\oplus V,W}(W) = 0$. Observe that $\delta_{M,N} - \delta_{M\oplus V,W} = \delta_\Phi$ and, from Lemma 2.9, $\delta_\Phi(N) > 0$. This leads to

$$\sum_{X\in\Gamma_A} \delta_{M\oplus V,W}(X) < \sum_{X\in\Gamma_A} \delta_{M,N}(X).$$

It follows from our inductive assumption that the modules $M \oplus V$ and W are isomorphic. Then the sequence Φ has the form

$$0 \to N \to V \oplus M \to V \to 0,$$

and this implies that $M <_{\deg} N$, by Proposition 3.4 in [13]. Applying Lemma 2.11, we get $\delta'_{M,N}(N) > 0$, and hence a contradiction. This finishes the proof. □

3.3. *Proof of Theorem 1.* Let M and N be A-modules with $M \le N$. We may assume that $M < N$. Let $r(X) = \min\{\delta_{M,N}(X), \mu(N,X)\}$, for any $X \in \Gamma_A$, and let $\mathcal{F}$ be the set of all vertices of Γ_A with $r(X) > 0$. The set $\mathcal{F}$ does not contain injective A-modules and is nonempty, by Lemma 3.2. Let $N' = \bigoplus_{X\in\mathcal{F}} X^{r(X)} = \bigoplus_{X\in\Gamma_A} X^{r(X)}$ and $N'' = \bigoplus_{X\in\Gamma_A} X^{\mu(N,X)-r(X)}$. Then $N = N' \oplus N''$. We define a new exact sequence without isomorphism

$$\Sigma : 0 \to \bigoplus_{X\in\mathcal{F}} X^{r(X)} \to \bigoplus_{X\in\mathcal{F}} E(X)^{r(X)} \to \bigoplus_{X\in\mathcal{F}} (\tau^- X)^{r(X)} \to 0.$$

Applying Lemma 2.5(i), we obtain $\delta_\Sigma(X) = r(X) \le \delta_{M,N}(X)$, for any $X \in \Gamma_A$. Consequently, $\delta_\Sigma \le \delta_{M,N}$ and, by Lemma 3.1, there is an exact sequence without isomorphism

$$\Phi : 0 \to N' \to W \to Z \to 0$$

with $\delta_\Sigma \le \delta_\Phi \le \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then $M \oplus Z \le N'' \oplus W$ and $\delta_{M\oplus Z,N''\oplus W}(W) = 0$. Let N_1 be any indecomposable direct summand of N''. Then $r(N_1) < \mu(N,N_1)$, and this leads to $\delta_\Sigma(N_1) = r(N_1) = \delta_{M,N}(N_1)$. Hence,

$$\delta_{M\oplus Z,N''\oplus W}(N_1) = \delta_{M,N}(N_1) - \delta_\Phi(N_1) = \delta_\Sigma(N_1) - \delta_\Phi(N_1) \le 0.$$

So, $\delta_{M\oplus Z,N''\oplus W}(N_1) = 0$. This implies that $\delta_{M\oplus Z,N''\oplus W}(N'') = 0$, and furthermore $\delta_{M\oplus Z,N''\oplus W}(N''\oplus W) = 0$. Hence, $M\oplus Z \simeq N''\oplus W$, by Lemma 3.2. Finally, the sequence Φ induces an exact sequence $0 \to N' \oplus N'' \to N'' \oplus W \to Z \to 0$, which has the form $0 \to N \to M \oplus Z \to Z \to 0$. In a similar way we obtain an exact sequence $0 \to Z' \to M \oplus Z' \to N \to 0$. □

Lemma 3.4. *Let M, N and X be A-modules such that $M < N$ and $X \in \Gamma_A$. Then we have:*

(i) *If $\delta'_{M,N}(X) > 0$, then there exist an indecomposable direct summand N_1 of N and a nonsplittable exact sequence $\Phi : 0 \to N_1 \to Y \to X \to 0$ without isomorphism such that $\delta_\Phi \le \delta_{M,N}$.*

(ii) *If $\delta_{M,N}(X) > 0$, then there exist an indecomposable direct summand N_1 of N and a nonsplittable exact sequence $\Phi : 0 \to X \to Y \to N_1 \to 0$ without isomorphism such that $\delta_\Phi \le \delta_{M,N}$.*

Proof. (i) Assume that $\delta'_{M,N}(X) > 0$. Applying Theorem 1 we get the exact sequence $\Sigma : 0 \to N \to M \oplus Z \to Z \to 0$, in $\operatorname{mod} A$. Further, applying Lemma 2.10(ii), we obtain a nonsplittable exact sequence $\Psi : 0 \to N \to W \to X \to 0$ with $\delta_\Psi \le \delta_\Sigma = \delta_{M,N}$. Then, by Lemma 2.9, there is an indecomposable direct summand

N_1 of N with $\delta_\Psi(N_1) > 0$. Finally, by Lemma 2.10(i), we obtain a nonsplittable exact sequence $\Phi : 0 \to N_1 \to Y \to X \to 0$ with $\delta_\Phi \leq \delta_\Psi \leq \delta_{M,N}$.

We obtain (ii) by duality. □

3.5. *Proof of Theorem 2.* Let B be an algebra and assume that $\mathrm{Ext}^1_B(X, X) = 0$ for any indecomposable B-module X. It is well-known that then B is representation-finite. Let M and N be two B-modules with $M \leq N$. We shall show that $M \leq_{\mathrm{ext}} N$. We proceed by induction on $[N, N] - [M, M] \geq 0$. If $[N, N] - [M, M] = 0$, then by Lemma 1.2 in [9], M is isomorphic to N. Hence, we may assume that $M < N$, and that M and N have no common nonzero direct summand. Take any indecomposable direct summand N_1 of N. Applying Lemma 2.5(ii), we obtain that $\delta_{M,N}(N_1) + \delta'_{M,N}(N_1) > 0$. Without loss of generality, we may assume that $\delta_{M,N}(N_1) > 0$. Now applying Lemma 3.4, we get a nonsplittable exact sequence

$$\Sigma : 0 \to N_1 \to Y \to N_2 \to 0$$

with $\delta_\Sigma \leq \delta_{M,N}$, for some A-module Y and some indecomposable direct summand N_2 of N. Since $\mathrm{Ext}^1_B(N_1, N_1) = 0$, the modules N_1 and N_2 are not isomorphic. Thus, $N = N_1 \oplus N_2 \oplus N_3$, for some A-module N_3. Moreover, $M \leq Y \oplus N_3 <_{\mathrm{ext}} N$. This implies that $[Y \oplus N_3, Y \oplus N_3] < [N, N]$, by Lemma 1.2 in [9]. Then

$$[Y \oplus N_3, Y \oplus N_3] - [M, M] < [N, N] - [M, M]$$

and $M \leq_{\mathrm{ext}} Y \oplus N_3$, by our inductive assumption. Finally, we obtain $M <_{\mathrm{ext}} N$, and this finishes the proof. □

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Faculty of Mathematics and Informatics, Nicholas Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: gzwara@mat.uni.torun.pl