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ON SWAN CONDUCTORS FOR BRAUER GROUPS OF CURVES OVER LOCAL FIELDS

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Abstract. For an element w of the Brauer group of a curve over a local field, we define the "Swan conductor" $\mathrm{sw}(w)$ of w, which measures the wildness of the ramification of w. We give a relation between $\mathrm{sw}(w)$ and Swan conductors for Brauer groups of henselian discrete valuation fields defined by Kato.

1. Introduction

Let k be a complete discrete valuation field with finite residue field F of characteristic p. Let O_k be the ring of integers in k. Let X be a projective smooth geometrically connected curve over k. There is a canonical pairing (cf. [4], [5], Section 9)

$$\langle , \rangle_X : \mathrm{Pic}(X) \times \mathrm{Br}(X) \to \mathbf{Q}/\mathbf{Z}, \tag{1}$$

described as follows: For a closed point $x \in X$ and an element $w \in \mathrm{Br}(X)$, we have the localization $w_x \in \mathrm{Br}(\kappa(x))$ of w at x. Since $\kappa(x)$ is a finite extension of k, we have $\mathrm{Cor}_{\kappa(x)/k}(w_x) \in \mathrm{Br}(k) \cong \mathbf{Q}/\mathbf{Z}$, where the last isomorphism is given by local class field theory. Then we define the pairing (1) by

$$\langle \sum_x n_x[x], w\rangle_X = \sum_x n_x \mathrm{Cor}_{\kappa(x)/k}(w_x) \in \mathbf{Q}/\mathbf{Z},$$

where $[x]$ denotes the class of x in $\mathrm{Pic}(X)$ and $n_x \in \mathbf{Z}$.

The pairing (1) induces an isomorphism

$$\mathrm{Br}(X) \xrightarrow{\cong} \mathrm{Hom}_c(\mathrm{Pic}(X), \mathbf{Q}/\mathbf{Z}).$$

Here Hom_c denotes the group of all continuous homomorphisms of finite order, and $\mathrm{Pic}(X)$ is endowed with a certain topology defined in [5], 9.4 (cf. Lemma 3.2). This result was first proven by Lichtenbaum, ignoring the p-primary part when $\mathrm{char}(k) = p > 0$, and more recently the general case was proven by Saito (cf. loc. cit.).

Taking a model of X over O_k, we can define a decreasing filtration $U^m \mathrm{Pic}(X) \subset \mathrm{Pic}(X)$ $(m > 0)$. For $w \in \mathrm{Br}(X)$, we define the *Swan conductor* $\mathrm{sw}(w)$ of w to be the minimal number m such that the map $\langle \cdot, w\rangle_X$ annihilates $U^{m+1} \mathrm{Pic}(X)$ (cf. Definition 3.1). This Swan conductor measures the ramification of w.

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On the other hand, for an element w of the Brauer group of a henselian discrete valuation field Λ, Kato defined the Swan conductor $\mathrm{sw}_\Lambda(w)$ of w, which again measures the ramification of w (cf. Definition 2.1). A certain description of those Swan conductors in terms of division algebras is given in [6] (cf. Theorem 2.2).

The following is the main theorem of this note:

Theorem 1.1. *Let $\mathfrak{X}$ be a regular model of X over O_k (cf. the beginning of Section 3). For each generic point $\eta \in Y = (\mathfrak{X} \otimes_{O_k} \bar{F})_{red}$, let K_η be the fraction field of the henselization of $O_{\mathfrak{X},\eta}$ and e_η the multiplicity of $\overline{\{\eta\}}$ in the divisor $\mathfrak{X} \otimes F$. Then, for $w \in \mathrm{Br}(X)$ we have*

$$\mathrm{sw}(w) = \sup\{\ [\mathrm{sw}_{K_\eta}(w_{K_\eta})/e_\eta] \mid \eta \text{ runs over the generic points of } Y\ \},$$

where w_{K_η} is the natural image of w in $\mathrm{Br}(K_\eta)$, and $[\ ]$ denotes the least integer function.

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2. Review on Swan conductors of Kato

In this section, we briefly recall Swan conductors for Brauer groups of henselian discrete valuation fields, which were defined by Kato (cf. [3], Proposition 6.5). Let Λ be a henselian discrete valuation field with residue field E such that $\mathrm{char}(E) = p > 0, [E : E^p] = p$, and $p\,\mathrm{Br}(E) \neq 0$. Cup products

$$H^1(\Lambda, \mathbf{Z}/m\mathbf{Z}(1)) \times H^2(\Lambda, \mathbf{Z}/m\mathbf{Z}(1)) \to H^3(\Lambda, \mathbf{Z}/m\mathbf{Z}(2)) \qquad (m > 0)$$

induce a pairing

$$\langle , \rangle_\Lambda : \Lambda^* \times \mathrm{Br}(\Lambda) \to H^3(\Lambda, \mathbf{Q}/\mathbf{Z}(2)). \tag{2}$$

Here, if $\mathrm{char}(\Lambda) = 0$, $\mathbf{Z}/m\mathbf{Z}(q)$ is defined to be the usual Tate twist of the constant sheaf $\mathbf{Z}/m\mathbf{Z}$. If $\mathrm{char}(\Lambda) = p$, write $m = p^s m'$ with $s \geq 0, p \nmid m'$. Then $\mathbf{Z}/m\mathbf{Z}(q)$ denotes the object $\mathbf{Z}/m'\mathbf{Z}(q) \oplus W_s\Omega^q_{\Lambda,\log}[-q]$ of the derived category of abelian sheaves on $\mathrm{Spec}(\Lambda)_{et}$, where $\mathbf{Z}/m'\mathbf{Z}(q)$ is defined to be the usual Tate twist and $W_s\Omega^q_{\Lambda,log}$ is the logarithmic part of the de Rham-Witt complex $W_s\Omega^q_\Lambda$. The group $H^3(\Lambda, \mathbf{Q}/\mathbf{Z}(2))$ is defined as the inductive limit of $H^3(\Lambda, \mathbf{Z}/m\mathbf{Z}(2))$.

Definition 2.1. For $w \in \mathrm{Br}(\Lambda)$, we define the Swan conductor $\mathrm{sw}_\Lambda(w)$ of w to be the non-negative integer

$$\inf\{m \mid \ker(\langle \cdot, w\rangle_\Lambda) \supset U_\Lambda^{(m+1)}\},$$

where $U_\Lambda^{(m)}$ is the m-th unit group of Λ.

A description of those Swan conductors in terms of division algebras is given by the following theorem, which was proven in [6].

Theorem 2.2. *Let Λ be as above and $w \in \mathrm{Br}(\Lambda)$. Let $\hat{\Lambda}$ be the completion of the strict henselization of Λ, and D the division algebra over $\hat{\Lambda}$ corresponding to the natural image $w_{\hat{\Lambda}}$ of w in $\mathrm{Br}(\hat{\Lambda})$.*

(i) The order of $w_{\hat{\Lambda}}$ *is equal to* $[D : \hat{\Lambda}]^{1/2}$ *which is equal to* p^n *for some non-negative integer* n.

(ii) For any subset S *of* D^*, *we write*

$$t_D(S) = \inf\{\mathrm{ord}_D(aba^{-1}b^{-1} - 1) \mid a, b \in S\},$$

where ord_D *denotes the normalized valuation on* D. *For* $j = 0, 1, \ldots, n-1$, *put*

$$t_j = \sup\{t_D({D'}^*) \mid D' \textit{ satisfies conditions below}\},$$
$$D' \quad \textit{is a division algebra},$$
$$\hat{\Lambda} \subset D' \subset D,$$
$$[D' : \textit{center of } D'] = p^{2j+2},$$
$$[\textit{center of } D' : \hat{\Lambda}] = p^{n-j-1}.$$

Then we have

$$\mathrm{sw}_\Lambda(w) = \frac{t_0}{p^{n-1}} + \sum_{j=1}^{n-1} \frac{(p-1)t_j}{p^{n-j}}.$$

Proof. Since the residue field of $\hat{\Lambda}$ is separably closed, (i) is deduced from [6], Proposition 2.1. (Note that the condition (*) in [6] is automatically satisfied.) By [3], Lemma 6.2, we have $\mathrm{sw}_\Lambda(w) = \mathrm{sw}_{\hat{\Lambda}}(w_{\hat{\Lambda}})$. (Though the Brauer group of the residue field of $\hat{\Lambda}$ is trivial, the definition of $\mathrm{sw}_{\hat{\Lambda}}(w_{\hat{\Lambda}})$ is given in [3].) Hence (ii) is deduced from [6], Theorem 5.1. □

3. Proof of Theorem 1.1

Let p, k, O_k, F and X be as in Section 1. Due to [1] and [2], there exists a scheme $\mathfrak{X}$ over O_k which satisfies the following property: $\mathfrak{X}$ is a two-dimensional regular proper flat shceme over O_k, $\mathfrak{X} \otimes_{O_k} k \cong X$ and $Y = (\mathfrak{X} \otimes_{O_k} F)_{\mathrm{red}}$ is a geometrically connected proper one-dimensional scheme over F whose irreducible components are all regular and which has ordinary double points as singularities at worst. Let $j : X \to \mathfrak{X}$ and $i : Y \to \mathfrak{X}$ be the inclusion morphisms.

We use the following conventions.

Let Y_0 (resp. Y_1) be the set of all closed (resp. generic) points of Y.

As in Theorem 1.1, for $\eta \in Y_1$, let K_η be the fraction field of the henselization of $O_{\mathfrak{X},\eta}$ and e_η the multiplicity of $\overline{\{\eta\}}$ in the divisor $\mathfrak{X} \otimes F$. Similarly, for a closed point $x \in X_0$, let K_x be the fraction field of the henselization of $O_{X,x}$.

Let $y \in Y_0$. Let A_y be the henselization of $O_{\mathfrak{X},y}$, $R_y = A_y \otimes_{O_k} k$, and K_y be the fraction field of A_y. Let Y_1^y denote the set of all height one prime ideals in A_y lying over some element of Y_1. (Note that the cardinality of Y_1^y is 1 or 2.) Similarly, let X_0^y denote the set of all closed points in X whose closure in $\mathfrak{X}$ includes y.

For $\eta \in Y_1$ and $y \in \overline{\{\eta\}} \cap Y_0$, there is a unique element η_y of Y_1^y lying over η. Let $O_{K_{\eta_y}}$ be the henselization of A_y at η_y, and K_{η_y} the fraction field of $O_{K_{\eta_y}}$.

Now we define a decreasing filtration on $\mathrm{Pic}(\mathfrak{X})$ and $\mathrm{Pic}(X)$. Fix a prime element π of k. For $m > 0$, let $\mathfrak{X}_m = \mathfrak{X} \otimes_{O_k} (O_k/\pi^m O_k)$, and let

$$U^m \operatorname{Pic}(\mathfrak{X}) = \ker(\operatorname{Pic}(\mathfrak{X}) \to \operatorname{Pic}(\mathfrak{X}_m)),$$
$$U^m \operatorname{Pic}(X) = \operatorname{Im}(U^m \operatorname{Pic}(\mathfrak{X}) \to \operatorname{Pic}(X)).$$

We define the Swan conductors of elements of $\mathrm{Br}(X)$:

Definition 3.1. For $w \in \mathrm{Br}(X)$, we define the Swan conductor $\mathrm{sw}(w)$ of w to be the non-negative integer

$$\inf\{m \mid \ker(\langle \cdot, w\rangle_X) \supset U^{m+1}\,\mathrm{Pic}(X)\}.$$

Now we begin the proof of Theorem 1.1. Let $T = H^1(Y_{et}, i^* Rj_* \mathbb{G}_m)$. We have a canonical isomorphism (cf. [5], 9.5)

$$T \cong \mathrm{Coker}(\bigoplus_{\eta \in Y_1} K_\eta^* \to \bigoplus_{y \in Y_0}((\bigoplus_{\eta_y \in Y_1^y} K_{\eta_y}^*)/R_y^*)\ ).$$

For each integer $m > 0$, let $U^m T$ be the image in T of

$$\bigoplus_{y \in Y_0} \bigoplus_{\eta_y \in Y_1^y} (1 + \pi^m O_{K_{\eta_y}}).$$

Consider the natural homomorphism

$$\psi : \mathrm{Pic}(X) \to T.$$

Lemma 3.2. *For each $m > 0$, $\psi(U^m\,\mathrm{Pic}(X)) \subset U^m T$ and we have*

$$\mathrm{Pic}(X)/U^m\,\mathrm{Pic}(X) \xrightarrow{\cong} T/U^m T.$$

Furthermore, the homomorphism

$$\psi^c : \mathrm{Pic}(X) \to T^c = \varprojlim T/U^m T$$

induced by ψ is a homeomorphism, when we consider T^c as a topological group by taking the image of $U^m T$ in T^c for $m > 0$ as a basis of neighborhoods at the origin. (Hence, $\{U^m\,\mathrm{Pic}(X)\}_{m>0}$ is a fundamental system of neighborhoods of $\mathrm{Pic}(X)$ *at the origin.)*

Proof. See [5] Lemma 9.8 and 9.10. □

For $y \in Y_0$ and $\eta_y \in Y_1^y$, there exists a canonical isomorphism

$$H^3(K_{\eta_y}, \mathbf{Q}/\mathbf{Z}(2)) \cong \mathbf{Q}/\mathbf{Z}, \tag{3}$$

which is given by two-dimensional local class field theory. Under this identification, we have for each $w \in \mathrm{Br}(X)$ the following diagram:

$$\begin{array}{ccc} \bigoplus_{y \in Y_0} \bigoplus_{\eta_y \in Y_1^y} K_{\eta_y}^* & \xrightarrow{\alpha} & \mathbf{Q}/\mathbf{Z} \\ {}^{\beta}\downarrow & & \| \\ \mathrm{Pic}(X) & \xrightarrow{\langle \cdot, w\rangle_X} & \mathbf{Q}/\mathbf{Z}. \end{array} \tag{4}$$

Here α is defined by

$$(a_{y,\eta_y}) \mapsto \sum_{y \in Y_0} \sum_{\eta_y \in Y_1^y} \langle a_{y,\eta_y}, w_{K_{\eta_y}} \rangle_{K_{\eta_y}},$$

where $w_{K_{\eta_y}}$ is the natural image of w in $\mathrm{Br}(K_{\eta_y})$, and β is induced by ψ^{c-1}. This diagram is known to be anti-commutative (cf. [5] p. 411). We will briefly review the proof of this fact at the end of this section.

By Lemma 3.2, we have

$$\beta(\bigoplus_{y \in Y_0} \bigoplus_{\eta_y \in Y_1^y} (1 + \pi^m O_{K_{\eta_y}})) = U^m\,\mathrm{Pic}(X) \qquad (m > 0).$$

This shows that

$$\mathrm{sw}(w) = \sup\{\ [\mathrm{sw}_{K_{\eta_y}}(w_{K_{\eta_y}})/e_\eta] \mid \eta \in Y_1, y \in \bar{\{\eta\}} \cap Y_0\},$$

where $\eta_y \in Y_1^y$ is the unique element lying over η. By [3] Lemma 6.2, if $\eta \in Y_1$, for any $y \in \bar{\{\eta\}} \cap Y_0$ we have

$$\mathrm{sw}_{K_{\eta_y}}(w_{K_{\eta_y}}) = \mathrm{sw}_{K_\eta}(w_{K_\eta}),$$

and Theorem 1.1 follows.

From now on, we recall the proof of the anti-commutativity of (4). This is deduced from the following three facts and the definitions of (1) and (2). First, we have an explicit description of ψ (cf. [5], Section 7). For each $y \in Y_0$, we have a composite map

$$\psi_y : \bigoplus_{x\in X_0^y} \mathbf{Z} \longrightarrow \mathrm{Coker}[K_y^* \xrightarrow{\gamma} (\ (\bigoplus_{x\in X_0^y} \mathbf{Z}) \oplus (\bigoplus_{\eta_y\in Y_1^y} K_{\eta_y}^*)\ )] \xleftarrow{\cong} (\bigoplus_{\eta_y\in Y_1^y} K_{\eta_y}^*)/R_y^*.$$

Here γ is defined to be the sum of the natural maps $K_y^* \to K_{\eta_y}^*$ $(\eta_y \in Y_1^y)$ and the composite maps

$$K_y^* \to K_x^* \xrightarrow{\mathrm{ord}_{K_x}} \mathbf{Z} \qquad (x \in X_0^y),$$

where ord_{K_x} is the normalized valuation on K_x. Then the map ψ is equal to the map induced by $\bigoplus_{y\in Y_0} \psi_y$ on $\mathrm{Pic}(X)$.

Secondly, the reciprocity law (cf. [5], 2.9) shows the following fact. Similar to (3), there exists for each closed point $x \in X$ a canonical isomorphism

$$H^3(K_x, \mathbf{Q}/\mathbf{Z}(2)) \cong \mathbf{Q}/\mathbf{Z}, \tag{5}$$

which is again given by two-dimensional local class field theory. Under identifications (3) and (5), for $y \in Y_0$, $w \in \mathrm{Br}(K_y)$ and $a \in K_y^*$, we have an equation in $\mathbf{Q}/\mathbf{Z}$

$$\sum_{x\in X_0^y} \langle a, w_{K_x}\rangle_{K_x} + \sum_{\eta_y\in Y_1^y} \langle a, w_{K_{\eta_y}}\rangle_{K_{\eta_y}} = 0,$$

where w_{K_x} is the natural image of w in $\mathrm{Br}(K_x)$.

Finally, for a closed point $x \in X$ and $w \in \mathrm{Br}(X)$, we have a commutative diagram (cf. [5], 2.7)

$$\begin{array}{ccc} K_x^* & \xrightarrow{\langle\cdot, w_{K_x}\rangle_{K_x}} & \mathbf{Q}/\mathbf{Z} \\ {\scriptstyle \mathrm{ord}_{K_x}}\downarrow & & \| \\ \mathbf{Z} & \longrightarrow & \mathbf{Q}/\mathbf{Z}, \end{array}$$

where the lower horizontal arrow is defined by the localization $w_x \in \mathrm{Br}(\kappa(x)) \cong \mathbf{Q}/\mathbf{Z}$ of w at x. Here we again used the identification (5).

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