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A SIMPLE PROOF OF A CURIOUS CONGRUENCE BY SUN

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Abstract. In this note, we give a simple and elementary proof of the following curious congruence which was established by Zhi-Wei Sun:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}.$$

In [4], the following curious congruence for odd prime p was established by Zhi-Wei Sun:

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \pmod{p}. \tag{1}$$

The author's proof, using Pell sequences, is fairly complicated. In fact, a recent article [3] on congruence modulo p ends in the remark that "It seems unlikely that (1) can be proved with the simple approach that we have used here." In the present note, we give a simple and elementary proof of (1). Throughout, p denotes an odd prime.

First of all, it is well known (e.g. [1], [2]) that for $k = 0, 1, 2, \ldots, p-1$,

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}. \tag{2}$$

From (2) we get

$$\begin{aligned} \frac{2^{p-1}-1}{2} &= \frac{(1+1)^p - 2}{2p} = \frac{1}{2p} \sum_{k=1}^{p-1} \binom{p}{k} = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k} \binom{p-1}{k-1} \\ &\equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p}. \end{aligned} \tag{3}$$

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Let $\varepsilon = e^{\pi i/4}$. Then

$$
\begin{aligned}
(1+\varepsilon)^p + (1-\varepsilon)^p &= 2 + 2\sum_{\substack{1\le k\le p\\ k \text{ even}}} \binom{p}{k}\varepsilon^k \\
&= 2 + 2p\sum_{\substack{1\le k\le p\\ k \text{ even}}} \frac{1}{k}\binom{p-1}{k-1}\varepsilon^k \\
&\equiv 2 - 2p\sum_{\substack{1\le k\le p\\ k \text{ even}}} \frac{\varepsilon^k}{k} \pmod{p^2} \\
&= 2 - 2p\left(\sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^k}{4k} + i\sum_{k=1}^{[\frac{p+1}{4}]} \frac{(-1)^{k-1}}{4k-2}\right) \\
&= 2 - \frac{p}{2}\sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^k}{k} + ip\sum_{k=1}^{[\frac{p+1}{4}]} \frac{(-1)^k}{2k-1} \\
&= 2 - \frac{p}{2}A + ipB
\end{aligned}
\tag{4}
$$

where

$$A = \sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^k}{k} \quad \text{and} \quad B = \sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^k}{2k-1}.$$

Since $\overline{\varepsilon} = \varepsilon^{-1}$, taking modulus of both sides of (4) yields

$$
\begin{aligned}
4 - 2pA &\equiv \left(2 - \frac{p}{2}A\right)^2 + p^2B^2 \\
&\equiv 4 - 2pA \equiv ((1+\varepsilon)^p + (1-\varepsilon)^p)((1+\varepsilon^{-1})^p + (1-\varepsilon^{-1})^p) \\
&= (2+\varepsilon+\varepsilon^{-1})^p + (2-\varepsilon-\varepsilon^{-1})^p \\
&= (2+\sqrt{2})^p + (2-\sqrt{2})^p \\
&= 2^{p+1} + 2\sum_{\substack{1\le k\le p\\ k \text{ even}}} \binom{p}{k} 2^{p-k}(\sqrt{2})^k \\
&= 2^{p+1} + 2^{p+1}\sum_{k=1}^{(p-1)/2} \binom{p}{2k}\frac{1}{2^k} \\
&= 2^{p+1} + 2^p p\sum_{k=1}^{(p-1)/2} \frac{1}{k\cdot 2^k}\binom{p-1}{2k-1} \\
&\equiv 2^{p+1} - 2^p p\sum_{k=1}^{(p-1)/2} \frac{1}{k\cdot 2^k} \pmod{p^2}.
\end{aligned}
\tag{5}
$$

From (5) and (3) we obtain, since $2^{p-1} \equiv 1 \pmod{p}$,

$$\begin{aligned} A &\equiv -\frac{2^p - 2}{p} + 2^{p-1} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \\ &\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} + \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \pmod{p} \end{aligned}$$

and so

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} &\equiv -\sum_{k=1}^{p-1} \frac{(-1)^k}{k} + A = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{[\frac{p-1}{4}]} \frac{(-1)^k}{k} \\ &= \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{k=p-[\frac{p-1}{4}]}^{p-1} \frac{(-1)^{p-k}}{p-k} \\ &\equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} - \sum_{k=p-[\frac{p-1}{4}]}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p} \\ &= \sum_{k=1}^{[\frac{3p}{4}]} \frac{(-1)^{k-1}}{k} \end{aligned}$$

and (1) is proved.

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References

[1] Louis Comet, *Advanced Combinatorics*, D. Reidel Publishing Company, 1974.

[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fourth Edition, Clarendon Press, Oxford, 1960.

[3] Winfried Kohnen, *A simple congruence modulo p*, Amer. Math. Monthly **104** (1997), 444–445. MR **98e:**11004

[4] Zhi-Wei Sun, *A congruence for primes*, Proc. Amer. Math. Soc. **123** (1995), 1341–1346. MR **95f:**11003

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