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THE HOLOMORPHIC EXTENSION OF H^p-CR FUNCTIONS ON TUBE SUBMANIFOLDS

AL BOGGESS

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Abstract. We consider the set of CR functions on a connected tube submanifold of C^n satisfying a uniform bound on the L^p-norm in the tube direction. We show that all such CR functions holomorphically extend to H^p functions on the convex hull of the tube ($1 \le p \le \infty$). The H^p-norm of the extension is shown to be the same as the uniform L^p-norm in the tube direction of the CR function.

1. Definitions and main results

Recently, Boivin and Dwilewicz [BD] have generalized Bochner's Tube Theorem by showing that continuous CR functions on a tube-submanifold of C^n holomorphically extend to its convex hull. In this manuscript, we show that on a tube-submanifold, CR functions that satisfy a uniform L^p-estimate in the tube direction extend to an H^p function on the tube over the convex hull (here, $1 \le p \le \infty$). In addition, we show the H^p-norm on the convex hull of the holomorphic extension is bounded by the H^p-norm of the CR function.

We will be working in $C^n = R^n + iR^n$ with coordinates $x + iy,\ x \in R^n,\ y \in R^n$. Let N be a connected submanifold of R^n and let $M = N + iR^n$ be the (connected) tube over N. For $1 \le p \le \infty$, let $\mathrm{CR}^p(M)$ denote the space of CR functions (solutions to the tangential Cauchy-Riemann equations) on M which satisfy

$$\begin{aligned} ||f||_{p(M)}^p &= \sup_{x\in N} \int |f(x+iy)|^p \, dy \le A_p < \infty \ \text{ if } 1 \le p < \infty, \\ ||f||_{\infty(M)} &= \sup_{x\in N} ||f||_{L^\infty(T_x)} \le A_\infty < \infty \ \text{ if } p = \infty, \end{aligned}$$

where $T_x = \{x\} + iR^n$ (the tube over x).

If $1 \le p \le \infty$ and T is any tube of the form $T = U + iR^n$ with U an open set in R^n, then $H^p(T)$ will denote the usual space of H^p-functions on the tube T with the usual H^p-norm (defined as above with N replaced by U).

Our main theorem is the following.

Theorem 1 (Extension Theorem). *Suppose N is a connected submanifold of R^n, and let $M = N + iR^n$ be the tube over N. Let $\widehat{N}$ and $\widehat{M} = \widehat{N} + iR^n$ denote the interior of the convex hull of N and M, respectively. If $\widehat{M}$ is nonempty and if* $1 \le$

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$p \leq \infty$, then each element $f \in CR^p(M)$ extends to a unique element $F \in H^p(\widehat{M})$ with $||F||_{p(\widehat{M})} = ||f||_{p(M)}$. If $1 \leq p < \infty$, then for each $x_0 \in N$ and each closed, convex simplex $S \subset \widehat{N}$ with x_0 as a vertex

$$\lim_{x \mapsto x_0,\ x \in S} \int |F(x+iy) - f(x_0+iy)|^p \, dy \ = 0.$$

If $p = \infty$, then for each $x_0 \in N$ and almost every $y \in R^n$

$$\lim_{x \mapsto x_0,\ x \in S} F(x+iy) = f(x_0+iy).$$

Remark 1. If f is continuous and bounded, then the above convergence result for $p = \infty$ is true for *every* $y \in R^n$. This result is contained in [BD]. An earlier result along these lines is contained in [Kaz]. Microlocal results for the tube case are contained in [BTa], [K] and [T].

Remark 2. Since the extension, F, in this theorem is an element of $H^p(\widehat{M})$, all the boundary value results from H^p-Theory also apply. In particular, the pointwise, non-tangential boundary values within convex simplicies contained in $\widehat{M}$ exist at every $x \in N$ and almost every $y \in R^n$. For more details on these results see Stein and Weiss [SW].

A key result that is used in the proof of the above theorem is the following global H^p-version of Baouendi and Treves' Approximation Theorem for CR functions on tubes.

Theorem 2 (Approximation Theorem [BT])**.** *Let N be a connected submanifold of R^n and let $M = N + iR^n$ be the tube over N. If f is an element of $CR^p(M)$, then there exists a sequence of entire functions F_ϵ on C^n such that for each $x_0 \in N$*

$$\lim_{\epsilon \mapsto 0} \int |F_\epsilon(x_0+iy) - f(x_0+iy)|^p \, dy \ = 0 \quad \textit{for } 1 \leq p < \infty.$$

If $p = \infty$, then for each $x_0 \in N$, $\lim\limits_{\epsilon \mapsto 0} F_\epsilon(x_0+iy) = f(x_0+iy)$ for almost every $y \in R^n$. For $1 \leq p \leq \infty$, $||F_\epsilon||_{p(M)} \leq ||f||_{p(M)}$ for each $\epsilon > 0$.

Remark. If f is continuous, then the proof given below can be modified to show that the approximating sequence converges uniformly on the compact subsets of M.

2. Proof of the Approximation Theorem

The following proof is based on ideas set forth in [BT].

For any $z = x_0 + iy_0$, let

$$T_z = \{x_0 + it;\ t \in R^n\}$$

(i.e. the tube over the point x_0 passing through z). For $z = x_0 + iy_0 \in M$, let

$$G_\epsilon(z) = \frac{1}{\epsilon^n (\pi)^{n/2}} \int_{\zeta \in T_z} f(\zeta) e^{\epsilon^{-2}[\zeta - z]^2} \, d\zeta$$

where $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$ and where for $w \in C^n$, $[w]^2 = w_1^2 + \cdots + w_n^2$. Another description of G_ϵ is given by

$$G_\epsilon(z) = \frac{1}{\epsilon^n (\pi)^{n/2}} \int_{t \in R^n} f(x_0+it) e^{-\epsilon^{-2}|t-y_0|^2} \, dt.$$

Viewed this way, G_ϵ is the convolution of f in the tube-direction with an approximation to the identity (given by the spatial slices of the heat kernel). The following lemma can easily be established using standard techniques.

Lemma 1. *For each fixed* $x_0 \in N$

$$\lim_{\epsilon \mapsto 0} \int |G_\epsilon(x_0 + iy) - f(x_0 + iy)|^p \, dy = 0 \quad \textit{for } 1 \leq p < \infty.$$

If $p = \infty$*, then for each* $x_0 \in N$*,* $\lim_{\epsilon \mapsto 0} G_\epsilon(x_0 + iy) = f(x_0 + iy)$ *for almost every* $y \in R^n$*. For* $1 \leq p \leq \infty$*,* $||G_\epsilon||_{p(M)} \leq ||f||_{p(M)}$ *for each* $\epsilon > 0$*.*

Since the domain of integration defining $G_\epsilon(z)$ depends on z, this function is not, in general, analytic in z. However, if f is CR on M, then the domain of integration can be made independent of z as the next lemma shows. By a translation, assume that the origin 0 belongs to N.

Lemma 2. *For* $z \in C^n$*, let*

$$F_\epsilon(z) = \frac{1}{\epsilon^n (\pi)^{n/2}} \int_{\zeta \in T_0} f(\zeta) e^{\epsilon^{-2}[\zeta - z]^2} \, d\zeta.$$

For each $\epsilon > 0$*,* F_ϵ *is entire. If* f *is CR on* M*, then* $F_\epsilon(z) = G_\epsilon(z)$ *for* $z \in M$*.*

Proof. $F_\epsilon(z)$ is analytic in z in view of the following observations: the domain of integration, $T_0 = \{0 + it;\ t \in R^n\}$, is independent of z; the kernel $e^{\epsilon^{-2}[it - z]^2}$ is analytic in z and has exponential decay in t uniformly in z belonging to a compact set in C^n; and the function $t \mapsto f(0 + it)$ belongs to $L^p(R^n)$.

Now assume f is CR on M. We will show that $F_\epsilon(z) = G_\epsilon(z)$ for $z = x + iy \in M$. For $R > 0$, let $g_R(x + it)$ be a smooth function which is independent of x, equal to one on the set $\{t \in R^n;\ |t| \leq R\}$ and supported in the set $\{t \in R^n;\ |t| \leq R + 1\}$. Let

$$F_\epsilon^R(z) = \frac{1}{\epsilon^n (\pi)^{n/2}} \int_{\zeta \in T_0} g_R(\zeta) f(\zeta) e^{\epsilon^{-2}[\zeta - z]^2} \, d\zeta.$$

Define G_ϵ^R analogously (replacing T_0 with T_z). For each fixed $z \in M$, clearly $\lim_{R \mapsto \infty} F_\epsilon^R(z) = F_\epsilon(z)$ and $\lim_{R \mapsto \infty} G_\epsilon^R(z) = G_\epsilon(z)$.

Let $\gamma : [0, 1] \mapsto N$ be a smooth path which connects $0 = \gamma(0)$ to $x = \mathrm{Re}(z) = \gamma(1)$ (recall, by assumption that N is connected). Let

$$\tilde{T}_z = \{\gamma(u) + it;\ t \in R^n,\ 0 \leq u \leq 1\}.$$

The (manifold) boundary of $\tilde{T}_z$ is $T_z - T_0$. So by Stokes theorem

$$F_\epsilon^R(z) = G_\epsilon^R(z) + \frac{1}{\epsilon^n (\pi)^{n/2}} \int_{\zeta \in \tilde{T}_z} d_\zeta \{ g_R(\zeta) f(\zeta) e^{\epsilon^{-2}[\zeta - z]^2} \, d\zeta \}.$$

We must show the integral on the right converges to zero as $R \mapsto \infty$. In view of the presence of $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_n$, the d_ζ reduces to $\overline{\partial}_\zeta$ in the last integral. Since f is CR, the $\overline{\partial}_\zeta$ only applies to $g_R(\zeta)$ which has support in the set $R \leq |\mathrm{Im}\zeta| \leq R+1$. In view of the exponential decay of $e^{\epsilon^{-2}[\zeta - z]^2}$ as $|\mathrm{Im}\zeta| \mapsto \infty$ and the fact that $f(x+it)$ is an L^p-function in t, the above integral on the right converges to zero as $R \mapsto \infty$. This completes the proof of the lemma and hence the proof of the Approximation Theorem.

Technically speaking, the proof of the above lemma assumes that f is continuously differentiable for the Stokes theorem step. However, Stokes' theorem applies to currents (in fact Stokes' theorem becomes the definition of the exterior derivative of a current; see [B] for more details) and so the above argument can be dualized and applied to our context where f is assumed to be a distribution given by a locally integrable function with a uniform bound on its L^p norm over tube-slices.

3. Proof of the Extension Theorem

Suppose f is an element of $\mathrm{CR}^p(M)$. To extend f to an analytic function in $\widehat{M}$, the interior of the convex hull of M, we first show that this set can be realized as the set of centers of analytic discs with boundaries in M. Then we show, by a subaveraging technique on the boundaries of these discs, that the sequence of entire functions that converges to f on M (from the Approximation Theorem) also converges uniformly on compact subsets of $\widehat{M}$. We carry out this outline in a series of lemmas.

An *analytic disc* is an analytic map $A : D = \{\zeta \in C;\ |\zeta| < 1\} \mapsto C^n$ with boundary values $A|_{\{|\zeta|=1\}}$ in $L^2(\{|\zeta| = 1\})$.

Lemma 3. *Suppose $e_0, \dots, e_m$ are vectors in N that span a convex simplex S with nonempty interior in R^n, $(m \geq n)$. Then, each point $z = x + iy$ with $x \in S$ and $y \in R^n$ can be realized as the center of an analytic disc, $z = A(\zeta = 0)$, whose boundary is contained in M. If*

$$x = \sum_{j=0}^{m} \lambda_j e_j \in S \qquad \text{with } \lambda_j \geq 0 \text{ and } \sum_j \lambda_j = 1$$

then the boundary of the analytic disc $A(\cdot) = A(\lambda, y)(\cdot)$ depends continuously on $\lambda = (\lambda_0, \dots, \lambda_m)$ and y in the $L^2(\{|\zeta| = 1\})$-norm.

Proof. To establish this lemma, we will specify the desired analytic disc $A = u+iv : D \mapsto C^n$, by specifying A on the boundary $\{e^{2\pi i t};\ 0 \leq t < 1\}$ which we identify with the unit interval $I = [0, 1)$. Partition the unit interval I into a disjoint union of intervals, I_j, of length λ_j, $j = 0, \dots, m$ where $I_0 = [0, \lambda_0)$, $I_1 = [\lambda_0, \lambda_0 + \lambda_1)$, etc. Let χ_{I_j} be the characteristic function on the interval I_j (one on I_j, zero everywhere else). Define $u : I \mapsto N$ by

$$u(t) = \sum_{j=0}^{m} e_j \chi_{I_j}(t).$$

Since the length of I_j is λ_j, $u = u(\lambda)$ depends continuously on $\lambda = (\lambda_0, \dots, \lambda_m)$ in the $L^2(I)$-norm.

For $y \in R^n$, let

$$v = v(\lambda, y) = T(u(\lambda)) + y$$

where T is the Hilbert transform. Since $T : L^2(I) \mapsto L^2(I)$ is continuous, $v(\lambda, y)$ depends continuously on λ and y in the $L^2(I)$-norm. Note, however, that T is not continuous in the sup-norm and so even though u is bounded, Tu is unbounded (in fact Tu grows logarithmically at the endpoints of the I_j, where u is discontinuous).

Now let $A(\lambda, y)(e^{2\pi i t}) = u(\lambda)(t) + iv(\lambda, y)(t)$. $A(\lambda, y)(\cdot)$ extends analytically to the unit disc D (by the definition of the Hilbert transform). Its boundary lies in M since $\mathrm{Re} A = u$ takes values in N. We claim $A(\zeta = 0) = x + iy$, where $x = \sum_j \lambda_j e_j$.

Since $v = T(u) + y$ and since the Hilbert transform produces the unique harmonic conjugate which vanishes at the origin, clearly $\mathrm{Im}(A)(\zeta = 0) = y$. The real part, $\mathrm{Re}A(0)$, is given by averaging its boundary values.

$$\begin{aligned}\mathrm{Re}(A)(\zeta = 0) &= \int_0^1 u(\lambda)(t)\,dt \\ &= \sum_{j=0}^{m} \int_0^1 e_j \chi_{I_j}\,dt \\ &= \sum_{j=0}^{m} \lambda_j e_j \\ &= x.\end{aligned}$$

This completes the proof of the lemma.

We wish to show that the sequence of entire functions F_ϵ, which converges to our given CR function f on M (in the L^p-norm on tube slices) also converges uniformly on a neighborhood of each point $z_0 = x_0 + iy_0$ with $y_0 \in R^n$ and x_0 in the interior of the convex hull of N. The next lemma contains the key subaveraging step. For a point $x \in R^n$, let $B(x, r)$ denote the open ball in R^n centered at x of radius r.

Lemma 4. *Suppose F is analytic on $\widehat{M}$ (the interior of the convex hull of M) and continuous on $\overline{\widehat{M}}$. Let $1 \le p < \infty$. For a given $z_0 = x_0 + iy_0 \in \widehat{M}$, there exists an $r > 0$, and a constant $C = C(p, r)$ (depending only on r and p) such that for each $\tilde{z} = \tilde{x} + i\tilde{y} \in B(x_0, r) + iB(y_0, r)$*

$$|F(\tilde{z})| \le C \int_0^1 \int_{\lambda \in S} \int_{|y-y_0| \le 2r} |F(A(\lambda, y)(e^{2\pi i t}))|^p \,dy\,d\lambda\,dt$$

Proof. Fix $x_0 \in \widehat{N}$. Choose a convex simplex S with vertices $e_0, \ldots, e_m \in N$ so that x_0 belongs to the nonempty interior of S. S contains a ball, of radius $2r > 0$ in R^n about x_0. F is analytic, and so for $\tilde{z} = \tilde{x} + i\tilde{y} \in B(x_0, r) + iB(y_0, r)$

$$\begin{aligned}|F(\tilde{z})| &\le C(r) \int_{|x-x_0| \le 2r} \int_{|y-y_0| \le 2r} |F(x + iy)|\,dy\,dx \\ &\le C(p, r) \left(\int_{|x-x_0| \le 2r} \int_{|y-y_0| \le 2r} |F(x + iy)|^p \,dy\,dx \right)^{1/p}\end{aligned}$$

where the last inequality follows from Hölder's inequality.

Let $A = A(\lambda, y)$ be the analytic disc given in Lemma 3. The map

$$(\lambda, y) = (\lambda_0, \ldots, \lambda_m, y) \mapsto A(\lambda, y)(\zeta = 0) = \sum_{j=0}^{m} \lambda_j e_j + iy \in S + iR^n$$

is a linear map whose image contains all of $S + iR^n$ which in turn contains

$$\{|x - x_0| \le 2r\} + iR^n.$$

So

$$|F(\tilde{z})|^p \le C \int_{\lambda \in S} \int_{|y-y_0| \le 2r} |F(A(\lambda, y)(\zeta = 0))|^p \,dyd\lambda$$

for all $\tilde{z} \in B(x_0, r) + iB(y_0, r)$.

The proof of the lemma is now completed by using the following inequality which is a consequence of the fact that $|F(A(\lambda, y)(\zeta))|^p$ is subharmonic in ζ for $|\zeta| < 1$ and continuous up to $|\zeta| = 1$

$$|F(A(\lambda, y)(\zeta = 0))|^p \leq \int_0^1 |F(A(\lambda, y)(e^{2\pi it}))|^p \, dt.$$

Lemma 5. *For $1 \leq p \leq \infty$, the sequence F_ϵ from the Approximation Theorem converges uniformly on the compact subsets of $\widehat{M}$ (the interior of the convex hull of the tube M) to an analytic function F with $||F||_{p(\widehat{M})} \leq ||f||_{p(M)}$. In addition, for each $x \in \widehat{N}$*

$$\lim_{\epsilon \mapsto 0} \int_{y \in R^n} |F_\epsilon(x + iy) - F(x + iy)|^p = 0 \quad \text{if } 1 \leq p < \infty. \tag{1}$$

Proof. First assume $1 \leq p < \infty$. Choose any $z_0 = x_0 + iy_0 \in \widehat{M}$. Applying Lemma 4 to the entire function $F_{\epsilon_1} - F_{\epsilon_2}$, yields

$$|(F_{\epsilon_1} - F_{\epsilon_2})(\tilde{z})| \leq C \int_0^1 \int_{\lambda \in S} \int_{|y - y_0| \leq 2r} |(F_{\epsilon_1} - F_{\epsilon_2})(A(\lambda, y)(e^{2\pi it}))|^p \, dy d\lambda dt$$

for $\tilde{z} \in B(x_0, r) + iB(y_0, r)$. The boundary of the real part of the analytic disc A constructed in Lemma 3 is $u(\lambda)(t) = \sum_{j=0}^m e_j \chi_{I_j}$. The imaginary part of A is $T(u(\lambda)) + y$, which is finite everywhere on $[0, 1)$ except the endpoints of the intervals I_j. Therefore for $\tilde{z} \in B(x_0, r) + iB(y_0, r)$

$$\begin{aligned} &|(F_{\epsilon_1} - F_{\epsilon_2})(\tilde{z})| \\ &\quad \leq C \sum_{j=0}^m \int_{\lambda \in S} \int_{t \in I_j} \int_{|y - y_0| \leq 2r} |(F_{\epsilon_1} - F_{\epsilon_2})(e_j + i(T(u(\lambda))(t) + y))|^p \, dy d\lambda dt \qquad (2) \\ &\quad \leq C \sum_{j=0}^m \int_{\lambda \in S} \lambda_j ||F_{\epsilon_1} - F_{\epsilon_2}||^p_{L^p(T_{e_j})} \, d\lambda \qquad (3) \end{aligned}$$

where T_{e_j} is the tube over e_j. By the Approximation Theorem, the L^p norm of $F_\epsilon - f$ over T_{e_j} converges to zero as $\epsilon \mapsto 0$. So the right side converges to zero as $\epsilon_1, \ \epsilon_2 \mapsto 0$. Therefore $F_\epsilon(z)$ is uniformly Cauchy on $B(x_0, r) + iB(y_0, r)$. Since $x_0 \in \widehat{N}$ was arbitrarily chosen, we conclude that F_ϵ converges uniformly to an analytic function F defined on $\widehat{M}$.

To prove estimate (1), fix any $x \in \widehat{N}$. Choose S and $\lambda = (\lambda_0, \ldots, \lambda_m) \in S$ as in the proof of Lemma 4 with $A(\lambda, y)(\zeta = 0) = x + iy$. By subaveraging over the boundary of the disc A:

$$|(F_\epsilon - F_\delta)(x + iy)|^p \leq \int_0^1 |(F_\epsilon - F_\delta)(A(\lambda, y)(e^{2\pi it}))|^p \, dt \tag{4}$$

and then integrating y:

$$\begin{aligned}\int_{y\in R^n} &|(F_\epsilon - F_\delta)(x+iy)|^p \, dy \\ &\leq \sum_{j=0}^{m} \int_{t\in I_j} \int_{y\in R^n} |(F_\epsilon - F_\delta)(e_j + i(T(u(\lambda))(t)+y))|^p \, dy \, dt \\ &\leq \sum_{j=0}^{m} \lambda_j ||F_\epsilon - F_\delta||^p_{L^p(T_{e_j})}.\end{aligned}$$

Equation (1) is now established by letting $\delta \mapsto 0$ and then $\epsilon \mapsto 0$ and by using the Approximation Theorem.

The estimate on $||F||_{p(M)}$ is established in a similar manner by first showing

$$|F_\epsilon(x+iy)|^p \leq \int_0^1 |F_\epsilon(A(\lambda, y)(e^{2\pi i t}))|^p \, dt \tag{5}$$

and then integrating y

$$\int_{y\in R^n} |F_\epsilon(x+iy)|^p \, dy \ \leq \ \sum_{j=0}^{m} \int_{t\in I_j} \int_{y\in R^n} |F_\epsilon(e_j + i(T(u(\lambda))(t)+y))|^p \, dy \, dt.$$

After taking limits as $\epsilon \mapsto 0$, the above inequality holds with F on the left (in view of (1)) and f on the right (by the Approximation Theorem). The right side is then dominated by

$$\sum_{j=0}^{m} \lambda_j ||f||^p_{p(M)} = ||f||^p_{p(M)}$$

(since $\sum_j \lambda_j = 1$), as desired.

If $p = \infty$, then we use (2) with $p = 1$. The integrand on the right side of (2) is dominated by $2||f||_{\infty(M)} < \infty$ by Lemmas 1 and 2. The domain of integration on the right side of (2) is bounded. In view of the Approximation Theorem (for the case $p = \infty$) and the Dominated Convergence Theorem, we conclude that F_ϵ is uniformly Cauchy on $B(x_0, r) + iB(y_0, r)$. The estimate $||F||_{p(\widehat{M})} = ||f||_{p(M)}$ for $p = \infty$ follows easily from (5) with $p = 1$, and the inequality $||F_\epsilon||_{\infty(M)} \leq ||f||_{\infty(M)}$ (Lemmas 1 and 2).

The final step in the proof of the main theorem is the following lemma.

Lemma 6. *Suppose S is the convex hull of the vertices $e_0, \ldots, e_m \in N$ and suppose the interior of S is nonempty. Then*

$$\begin{aligned}\lim_{x\mapsto e_0,\ x\in S} \int_{y\in R^n} |F(x+iy) - f(e_0+iy)|^p \, dy &= 0 \qquad \text{if } 1 \leq p < \infty, \\ \lim_{x\mapsto e_0,\ x\in S} F(x+iy) &= f(e_0+iy) \quad \text{if } p = \infty.\end{aligned}$$

Since S and $e_0 \in N$ are arbitrarily chosen, this lemma will complete the proof of our main theorem.

Proof. First assume $1 \le p < \infty$. The arguments in the proof of the previous lemma (see (4) or (5)) yield the following estimate:

$$\begin{aligned}
&\int_{y\in R^n} |F_\epsilon(x+iy) - F_\epsilon(e_0+iy)|^p \, dy \\
\le\ & \sum_{j=0}^{m} \int_{y\in R^n} \int_{t\in I_j} |F_\epsilon(e_j + i(T(u(\lambda))(t)+y)) - F_\epsilon(e_0+iy)|^p \, dtdy
\end{aligned}$$

for each $x \in \widehat{N}$. In view of (1) and the Approximation Theorem, the above estimate holds in the limit as $\epsilon \mapsto 0$. After separating the integral over I_0 on the right, we obtain

$$\begin{aligned}
&\int_{y\in R^n} |F(x+iy) - f(e_0+iy)|^p \, dy \\
&\le \int_{y\in R^n} \int_{t\in I_0} |f(e_0 + i(T(u(\lambda)(t)+y)) - f(e_0+iy)|^p \, dy \\
&+2\sum_{j=1}^{m} \lambda_j ||f||^p_{p(M)}.
\end{aligned}$$

We will let $x = \lambda_0 e_0 + \sum_{j=1}^m \lambda_j e_j$ approach e_0 by letting $\lambda_0 \mapsto 1$ and $\sum_{j=1}^m \lambda_j \mapsto 0$. The sum on the right clearly converges to zero as $x \mapsto e_0$. Also, $u(\lambda) \mapsto e_0$, and hence $T(u(\lambda)) \mapsto T(e_0) = 0$ in $L^2([0,1))$ as $\lambda_0 \mapsto 1$. Since $u(\lambda) = e_0$ on $I_0 = [0, \lambda_0)$ and the kernel for the Hilbert transform has diagonal singularities, the following fact is true: for each fixed $0 < \eta < 1$, $|T(u(\lambda))| \mapsto 0$ *uniformly* on $[0, \eta]$ as $\lambda_0 \mapsto 1$.

The integral (over I_0) on the right side of the last inequality can now be split into two: one over $I_0 \cap [0,\eta]$ and the other over $I_0 \cap (\eta, 1]$. The integral over $I_0 \cap (\eta, 1]$ is dominated by $2(1-\eta)||f||_{p(M)}$ which can be made as small as desired by choosing η close to 1. The integral over $[0,\eta]$ converges to zero since $T(u(\lambda)) \mapsto 0$ uniformly on $[0,\eta]$ and because small translates of the L^p function $y \to f(e_0+iy)$ are close (in L^p-norm) to the function itself.

Thus, $\int_{y\in R^n} |F(x+iy) - f(e_0+iy)|^p \, dy \mapsto 0$ as $x \mapsto e_0$. This completes the proof of the lemma and of our Extension Theorem for the case $1 \le p < \infty$.

For the case $p = \infty$, the same arguments as above can be used to show that $F(x+iy)$ converges to $f(e_0+iy)$ weakly in $y \in R^n$ as $x \mapsto e_0$, i.e.

$$\lim_{x\mapsto e_0,\ x\in S} \int_{y\in R^n} |F(x+iy) - f(e_0+iy)| g(y) \, dy = 0$$

for each smooth, compactly supported function g. $|F|$ is bounded on $\widehat{M}$ (true when $p = \infty$) and so $|F|$ is non-tangentially bounded. Therefore the non-tangential boundary limits of $F(x+iy)$ exist as $x \mapsto e_0$ for almost every $y \in R^n$ by standard H^p-Theory. This limit clearly must be $f(e_0+iy)$ in view of the weak limit mentioned above.

4. Uniqueness

Of course, the estimate $||F||_{p(\widehat{M})} = ||f||_{p(M)}$ implies that the H^p-extension of $f \in \mathrm{CR}^p(M)$ given in the Extension Theorem is unique. It is also easy to show that there is only one H^p-extension of a given element $f \in \mathrm{CR}^p(M)$ (regardless of whether or not the extension satisfies the estimate $||F||_{p(\widehat{M})} \le ||f||_{p(M)}$). Indeed, let x be a point in $\widehat{N}$ and let $A(\lambda, y)(\cdot)$ be the analytic disc given in Lemma 3. Write $A(\lambda, y)(\cdot)$ as $A_0(\lambda)(\cdot) + iy$ where A_0 is independent of y. The image of $A_0(\lambda)(\cdot)$ is contained in the tube over the complex simplex S. For any $F \in H^p(\widehat{M})$ and any $0 < r < 1$

$$\int_{y \in R^n} |F(x + iy)|\, dy \leq \int_{y \in R^n} \int_0^1 |F(A_0(\lambda)(re^{2\pi it}) + iy)|\, dt\, dy$$

by the Subaveraging Principle for subharmonic functions. The inner integral on the right side is a monotonically increasing function of r. Since the boundary of $A_0(\lambda)(\cdot)$ is contained in $M = N + iR^n$, we conclude (by the Monotone Convergence Theorem) that if F vanishes on M, then the right side converges to zero as r increases to 1.

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Department of Mathematics, Texas A & M University, College Station, Texas 77843

E-mail address: al.boggess@math.tamu.edu