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ON REAL QUADRATIC FUNCTION FIELDS OF CHOWLA TYPE WITH IDEAL CLASS NUMBER ONE

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ABSTRACT. Let $\mathbb{F}_q$ be the finite field with q elements, $(2\nmid q)$, $k=\mathbb{F}_q(x)$, $K=k(\sqrt{D})$ where $D=D(x)=A(x)^2+a$ is a square-free polynomial in $\mathbb{F}_q[x]$ with $\deg A(x)\geq 1$ and $a\in\mathbb{F}_q^*$. In this paper several equivalent conditions for the ideal class number $h(O_K)$ to be one are presented and all such quadratic function fields with $h(O_K)=1$ are determined.

1. Introduction

Let $d=a^2+1\geq 2$ be a square-free integer. R.A.Mollin [8] presented several equivalent conditions for the class number of real quadratic number field $K=\mathbb{Q}(\sqrt{d})$ to be one. S.Chowla conjectured that there are exactly 6 such fields with class number one. R.A.Mollin and H.C.Williams [9] proved this conjecture under the assumption of the Riemann hypothesis for $\zeta_K(s)$.

In this paper we will present an analogy of Mollin's sufficient conditions for ideal class number $h(O_K)$ of real quadratic function field K to be one. We will show that the quadratic function field $K=k(\sqrt{D(x)})(k=\mathbb{F}_q(x),2\nmid q)$ satisfying these conditions has to be of Chowla type: $D(x)=A(x)^2+a$ where $A(x)\in\mathbb{F}_q[x]$ and $a\in\mathbb{F}_q^*$. Since the Riemann hypothesis for function fields is true (A.Weil's theorem), we can determine all such quadratic function fields K with $h(O_K)=1$.

2. Preliminary lemmas

A systematic research on quadratic the function fields was initiated by E.Artin [1] in 1924, who gave the analytic formula for the class number (see section 4) and made a small table of class numbers. Let $k=\mathbb{F}_q(x)$ be the rational function field $(2\nmid q)$, $D=D(x)$ a square-free polynimial in $\mathbb{F}_q[x]$ with $\deg D\geq 1$, $\operatorname{sgn}D$ the leading coefficient of the polynomial $D(x)$.Without loss of generality we can assume $\operatorname{sgn}D=1$ or g where g is a fixed generator of cyclic group $\mathbb{F}_q^*$. The quadratic function field $K=k(\sqrt{D})$ is called (by E.Artin) real if $2\mid\deg D$ and $\operatorname{sgn}D=1$ since $\sqrt{D}\in k_\infty=\mathbb{F}_q((\frac{1}{x}))$ (the completion of k at the infinite prime divisor $\infty=(\frac{1}{x})$). Otherwise K is called imaginary.

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Let O_K be the integral closure of $\mathbb{F}_q[x]$ in K, $h(O_K)$ the ideal class number of O_K. J.R.C.Leitzel, M.L.Madan, C.S.Queen and R.E.MacRae [5, 6, 7] determined all imaginary quadratic function fields K (even for the case $2|q$) with $h(O_K) = 1$. For the real case, we can ask the following question as an analogy of a Gauss conjecture: Are there infinitely many of real quadratic function fields K with $h(O_K) = 1$ for any fixed q?

Let $K = k(\sqrt{D})$ be a real function field, $\deg D = 2d \geq 2$. The completion of $k = \mathbb{F}_q(x)$ at $\infty = (\frac{1}{x})$ is the power series field $k_\infty = \mathbb{F}_q((\frac{1}{x}))$. Each element $0 \neq a \in k_\infty$ has the unique $(\frac{1}{x})$-adic expansion

$$a = \sum_{i=m}^{\infty} c_i \left(\frac{1}{x}\right)^i, \qquad c_i \in \mathbb{F}_q, \qquad c_m \neq 0.$$

Let $v_\infty(a) = m$, and $v_\infty(0) = \infty . v_\infty$ is an extension of $(\frac{1}{x})$-adic exponential valuation of k. Since $\sqrt{D} \in k_\infty$, K is a subfield of k_∞, and the restriction of v_∞ to K is an exponential valuation of K. The galois group Gal $(K/k) = \{1, \sigma\}$ where $\sigma(A + B\sqrt{D}) = A - B\sqrt{D}$ (A,B$\in$ k). We know that $O_K = \mathbb{F}_q[x] \oplus \mathbb{F}_q[x]\sqrt{D}$, and the unit group U_K of O_K is $\mathbb{F}_q^* \times \langle\varepsilon\rangle$ where ε is a generator of the free part of U_K. Let $N = N_{K/k}$ be the norm mapping for K/k. Since $N(\alpha\varepsilon) = \alpha^2 N(\varepsilon)$ for $\alpha \in \mathbb{F}_q^*$,we can assume $N(\varepsilon) = 1$ or g. Since $N(\varepsilon) = \varepsilon \cdot \sigma(\varepsilon) \in \mathbb{F}_q^*$, we have $0 = v_\infty(N(\varepsilon)) = v_\infty(\varepsilon) + v_\infty(\sigma(\varepsilon))$, and ε is determined by the condition $v_\infty(\varepsilon) < 0$ up to factor (± 1). We call this ε the fundamental unit of K.

At the first step, we give a criterion for $h(O_K) = 1$. For each ideal $\mathfrak{a} \neq (0)$, the order of the finite quotient ring $O_K/\mathfrak{a}$ is a *q-th*-power. If $|O_K/\mathfrak{a}| = q^m$, the degree of $\mathfrak{a}$ is defined by $\deg \mathfrak{a} = m \geq 1$. In [4] we showed that the Minkowski constant for real quadratic function field $K = k(\sqrt{D})$ is $d - 1$ which means that we have

Lemma 2.1. *Let $K = k(\sqrt{D})$ be a real quadratic function field,* $\deg D = 2d \geq 2$*.Then each fractional ideal class of O_K contains an integral ideal $\mathfrak{a}(\subseteq O_K)$ with* $\deg \mathfrak{a} \leq d - 1$*. Therefore $h(O_K) = 1$ if and only if all prime ideals of O_K with degree $\leq d - 1$ are principal.*

Next, we introduce a result of Xianke Zhang [10], who determined the 2-rank of the ideal class group of K by calculating the number of ambiguous ideal classes.

Lemma 2.2. *Let $K = k(\sqrt{D})$ be a real quadratic function field, $C(K)$ the ideal class group of O_K,$r = \dim_{\mathbb{F}_2} C(K)/C(K)^2$ the 2-rank of $C(K)$, m the number of monic irreducible polynomial factors of $D = D(x)$. Then $r = m - 2$ if D has an irreducible factor with odd degree, $r = m - 1$ otherwise. Therefore $2 \nmid h(O_K) = |C_K|$ if and only if*

(I) D is irreducible, or

(II) $D = P_1 P_2$ where P_1 and P_2 are irreducible polynomials with odd degree.

Remark 2.3. Let ε be the fundamental unit of K. For the case (I) of lemma 2.2, E.Artin [1] proved $N(\varepsilon) = g$. For the case (II) we have $N(\varepsilon) = 1$.

Proof. Let $\varepsilon = A + B\sqrt{D}(A, B \in \mathbb{F}_q[x])$, P be an irreducible factor of D with odd degree. If $g = N(\varepsilon) = A^2 - DB^2$, we have $A^2 \equiv g \pmod{P}$ and $(\frac{g}{P}) = 1$ where $(\frac{g}{P})$ is the Legendre symbol. But $2 \nmid \deg P$ implies $(\frac{g}{P}) = -1$, so we have contradiction (for more information on Legendre and Jacobi symbol in polynomial case, see section 4).

3. Real quadratic function fields of Chowla type

In this section we give an analogy of Mollin's result ([8], theorem 1) for real function field $K = k(\sqrt{D})$. At first we need some lemmas.

Definition 3.1. Let $E \in \mathbb{F}_q[x]$, $E \neq 0$. A solution $(X, Y) = (U, V)$ of the equation

$$(*) \qquad X^2 - DY^2 = E,$$

in $\mathbb{F}_q[x]$ is called trivial if $E = aM^2$, $a \in \mathbb{F}_q^*$, $M \in \mathbb{F}_q[x]$, and $M|U, M|V$.

Lemma 3.2. *Let $\varepsilon = A + B\sqrt{D}$ be the fundamental unit of $K.A, B \in \mathbb{F}_q[x]$. If the equation $(*)$ has a non-trivial solution in $\mathbb{F}_q[x]$, then $\deg E \geq \deg A - 2\deg B$.*

Proof. Let (U, V) be a non-trivial solution of the equation $(*)$, then $V \neq 0$. We can assume that (U, V) is a non-trivial solution with minimal $\deg V$. We have

$$\begin{aligned} N(\varepsilon)E &= N[(A + B\sqrt{D})(U \pm V\sqrt{D})] \\ &= (AU \pm DBV)^2 - D(BU \pm AV)^2. \end{aligned}$$

We claim that both of the solutions $(X, Y) = (AU \pm BDV, BU \pm AV)$ are non-trivial. If one of them is trivial, them $E = aM^2$,$a \in \mathbb{F}_q^*$ and

$$\begin{matrix} (1) \\ (2) \end{matrix} \qquad \left\{ \begin{aligned} AU + DBV &\equiv 0 \pmod{M} \\ BU + AV &\equiv 0 \pmod{M} \end{aligned} \right. \quad \text{or} \quad \left\{ \begin{aligned} AU - DBV &\equiv 0 \pmod{M} \\ BU - AV &\equiv 0 \pmod{M}. \end{aligned} \right.$$

But

$$(1) \cdot B - (2) \cdot A \Rightarrow A^2V - DB^2V \equiv 0 \pmod{M} \Rightarrow V \equiv 0 \pmod{M},$$

$$(1) \cdot A - (2) \cdot BD \Rightarrow A^2U - DB^2U \equiv 0 \pmod{M} \Rightarrow U \equiv 0 \pmod{M}.$$

Therefore the solution (U, V) is trivial. This contradiction implies that both of the $(X, Y) = (AU \pm DBV, BU \pm AV)$ are non-trivial solutions. Since $\deg V$ is minimal we know that

$$(3) \qquad \min\{\deg(BU + AV)),\ \deg(BU - AV)\} \geq \deg V.$$

On the other hand,

$$\begin{aligned} &(4) \qquad \deg E &&= \deg(U^2 - DV^2) = \deg(B^2U^2 - DB^2V^2) - 2\deg B \\ &(5) &&= \deg(B^2U^2 - A^2V^2 + V^2N(\varepsilon)) - 2\deg B. \end{aligned}$$

If $\deg BU > \deg AV$, then $\deg E = \deg(B^2U^2) - 2\deg B > \deg(A^2V^2) - 2\deg B > 2\deg A - 2\deg B$. If $\deg BU < \deg AV$, then $\deg A^2V^2 > \deg B^2U^2$, $\deg A^2V^2 > \deg(V^2N(\varepsilon))$, therefore $\deg E = \deg(A^2V^2) - 2\deg B \geq 2\deg A - 2\deg B$. Finally, if $\deg BU = \deg AV$, then $\max(\deg(BU + AV), \deg(BU - AV)) = \deg AV$. From (3) we know that $\deg(B^2U^2 - A^2V^2) \geq \deg AV^2$ and from (5) we have $\deg E \geq \deg AV^2 - 2\deg B \geq \deg A - 2\deg B$. Q.E.D.

Lemma 3.3. *Let f be a positive integer. The following two conditions are equivalent.*

(1) For each monic irreducible polynomial P in $\mathbb{F}_q[x]$ with $\deg P \leq f$, we have $(\frac{D}{P}) = -1$ (which means that P is inert in K).

(2) For each polynomial A and irreducible polynomial P in $\mathbb{F}_q[x]$ satisfying $\deg A < \deg P \leq f$ (we assume $\deg 0 = -\infty$) we have $A^2 - D \not\equiv 0 \pmod{P}$.

Proof. It is obvious.

Remark 3.4. The conditions in lemma 3.3 can be satisfied only for $f \leq d-1$ ($2d = \deg D$) since there exists a monic irreducible polynomial P such that $\deg P \leq d$ and $(\frac{D}{P}) \neq -1$.

Proof. D can always be expressed as $D = A^2 + B$ with $\deg A = d$ and $\deg B \leq d-1$. If $\deg B \geq 1$, we choose P as an irreducible factor of B, then $D \equiv A^2 (\text{mod } P)$ and $(\frac{D}{P}) \neq -1$. If $B = b \in \mathbb{F}_q^*$, it is easy to see that there exists $a \in \mathbb{F}_q$ such that $a^2 + b$ is a square in $\mathbb{F}_q$. Then $D = A^2 + b \equiv a^2 + b (\text{mod } A - a)$. We choose P as an irreducible factor of $A - a$. Then $D \equiv a^2 + b(\text{mod } P)$ and $(\frac{D}{P}) \neq -1$.

Now we come to our main result which shows that each condition in lemma 3.3 with $f = d-1$ is equivalent to $h(O_K) = 1$ and K is of Chowla type.

Theorem 3.5. *Let $K = k(\sqrt{D})$ be a real quadratic function field,* $\deg D = 2d \geq 2$, *$k = \mathbb{F}_q(x), 2 \not| q$. The following conditions are equivalent to each other.*

(1) For any monic irreducible polynomial P in $\mathbb{F}_q[x]$ with $\deg P \leq d-1$, we have $(\frac{D}{P}) = -1$.

(2) For any $A \in \mathbb{F}_q[x]$ and irreducible polynomial $P \in \mathbb{F}_q[x]$ satisfying $\deg A < \deg P \leq d-1$ we have $D - A^2 \not\equiv 0 (\text{mod } P)$.

(3) For any $A \in \mathbb{F}_q[x]$ with $\deg A \leq d-1$, $D - A^2$ is either irreducible or a product of two irreducible polynomials with degree d.

(4) $h(O_K) = 1$ and K is of Chowla type: $D = A^2 + b, A \in \mathbb{F}_q[x], b \in \mathbb{F}_q^*$.

Proof. (1)$\Longleftrightarrow$(2): By lemma 3.3.

(2)$\Longrightarrow$(3): If $\deg A \leq d-1$ and $D - A^2$ has an irreducible factor P with $\deg P \leq d-1$, we can assume that $\deg A < \deg P$ by replacing A if necessary by its least residue *mod* P. Therefore $\deg A < \deg P \leq d-1$ and $D - A^2 \equiv 0 (\text{mod } P)$ which contradicts (2).

(3)$\Longrightarrow$(2): If $\deg A < \deg P \leq d-1$ and $D - A^2 \equiv 0(\text{mod } P)$,then $D - A^2$ has the irreducible factor P with $\deg P \leq d-1$ which contradicts (3).

(4)$\Longrightarrow$(1): We have $\varepsilon = A + \sqrt{D}$,$\deg A = d \geq 1$. If $\deg P \leq d-1$ and $(\frac{D}{P}) \neq -1$, then P either ramifies or splits in O_K. Thus we have a prime ideal $\mathfrak{p}$ in O_K such that $\mathfrak{p} \cdot \sigma(\mathfrak{p}) = P$. From $h(O_K) = 1$ we know that $\mathfrak{p}$ is principal: $\mathfrak{p} = (U + V\sqrt{D})$, therefore $\sigma(\mathfrak{p}) = (U - V\sqrt{D})$ and

$$U^2 - V^2 D = cP \qquad (c \in \mathbb{F}_q^*).$$

The solution $(X, Y) = (U, V)$ of the equation $X^2 - DY^2 = cP$ is non-trivial since $E = cP$ does not have the form aM^2. By lemma 3.2 we have $\deg P \geq \deg A - 0 = d$ which contradicts $\deg P \leq d-1$.

(1)$\Longrightarrow$(4): $h(O_K) = 1$ comes from lemma 2.1. Moreover, we have the expression $D = A^2 + B$ with $\deg A = d$ and $\deg B \leq d-1, B \neq 0$. If $\deg B \geq 1$,then B has an irreducible factor P with $\deg P \leq d-1$. We have $(\frac{D}{P}) \neq -1$ which contracicts (1). Therefore $B \in \mathbb{F}_q^*$ and K is of Chowla type. Q.E.D.

Remark 3.6. In fact, we can say more for D if $K = k(\sqrt{D})$ satisfies the conditions of theorem 3.5. From condition (4) we know $D = A^2 - a, a \in \mathbb{F}_q^*$ and $\deg A = d$. From condition (1), (2), or (3) we know that D has no irreducible factor P with $\deg P \leq d-1$. Therefore either D is irreducible (and a is not a square in $\mathbb{F}_q^*$) or $D = P_1 P_2$ where P_1 and P_2 are irreducible with $\deg P_1 = \deg P_2 = d$. Since $h(O_K) = 1$, the lemma 2.2 implies $2 \not| d$ in the case $D = P_1 P_2$. From remark 2.3 we know that $N(\varepsilon) = N(A+\sqrt{D}) = a$ is a square in $\mathbb{F}_q^*$. Therefore $a = b^2 (b \in \mathbb{F}_q^*)$ and $P_1, P_2 = A \pm b$. Thus $D = A^2 - a$ has very special property:

(I) $D = A^2 - a$ is irreducible and a is not a square in $\mathbb{F}_q^*$; or

(II) $D = (A+b)(A-b) = A^2 - b^2$ and $A \pm b$ are irreducible with odd degree d.

Suppose that $K = k(\sqrt{D})$ is a real quadratic function field and D satisfies the condition (I) or (II) of remark 3.6. If D does not satisfy the condition (1) of theorem 3.5, then $h(O_K) > 1$ and $2 \not| h(O_K)$ (lemma 2.2). The following theorem presents a better lower bound for $h(O_K)$.

Theorem 3.7. *Suppose that $K = k(\sqrt{D})$ is a real quadratic function field and D satisfies the condition (I) or (II) of remark 3.6. If there exists an irreducible P such that $\deg P \leq d-1$ and $(\frac{D}{P}) \neq -1$, then $h(O_K) \geq \langle \frac{d}{\deg P} \rangle$ where $< \alpha >$ denotes the smallest odd integer n such that $n \geq \alpha$.*

Proof. From the assumption and lemma 2.2 we known that $h(O_K)$ is odd. From $(\frac{D}{P}) \neq -1$ we know that $PO_K = \mathfrak{p} \cdot \sigma(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of O_K. Let n be the order of the ideal class $[\mathfrak{p}]$, then $2 \not| n | h(O_K)$ and $\mathfrak{p}^n$ is a principal ideal. Let $\mathfrak{p}^n = (U + V\sqrt{D}), U, V \in \mathbb{F}_q[x]$, then $U^2 - V^2 D = cP^n$,$c \in \mathbb{F}_q^*$. From $2 \not| n$ we know that $(X, Y) = (U, V)$ is a non-trivial solution of the equation $X^2 - DY^2 = cP^n$. Lemma 3.2 implies that $\deg P^n \geq d$. Therefore $h(O_K) \geq n \geq \frac{d}{\deg P}$. Since $2 \not| h(O_K)$ we know that $h(O_K) \geq \langle \frac{d}{\deg P} \rangle$. Q.E.D.

4. Determination of all real quadratic function fields of Chowla type with class number one

This task has essentially been done in [3] since the following theorem is proved by using the Weil theorem and the Riemann-Roch theorem.

Theorem 4.1 ([3]). *Suppose that $k = \mathbb{F}_q(x)$, $2 \not| q$, $K = k(\sqrt{D})$ is a real quadratic function field, $D = A^2 + a, a \in \mathbb{F}_q^*$, $\deg A = d \geq 1$. If $h(O_K) = 1$, then $q = 3$, $d \leq 4$; $q = 5$, $d \leq 2$; or $q \geq 7, d = 1$.*

Proof. We rewrite the proof here for the reader's convenience. The argument is taken from [7], p. 424. We know that $h(O_K)R_K = h(K)$ where $R_K = -v_\infty(\varepsilon) = -v_\infty(A + \sqrt{D}) = d$ (the regulator of K) and $h(K)$ is the divisor class number of K(=the order of the divisor class group of degree zero). The genus of K is $g_K = d - 1$.

Let $n = 2g_k - 1, \bar{k} = \mathbb{F}_{q^n}(x)$. Then $\bar{K} = \bar{k}(\sqrt{D})$ is a function field over $\mathbb{F}_{q^n}$ and $g_{\bar{K}} = g_K = d-1$. Let $\bar{N}_1$ be the number of prime divisors of $\bar{K}$ with degree 1. The Weil theorem implies that

$$\bar{N}_1 \geq q^n + 1 - 2g_{\bar{K}} \cdot q^{n/2}.$$

$\bar{K}/K$ is a constant extension of degree n. Each prime divisor of $\bar{K}$ with degree e is a product of (e, n) distinct prime devisors of K with degree $e/(e, n)$ (see [2], p. 164). Therefore the prime divisors of $\bar{K}$ with degree one come from the prime divisors of K with degree $e|n$. And for $e|n$, a prime divisor $\mathfrak{p}$ of K with degree e give $e(\leq n)$ prime divisors of $\bar{K}$ with degree one and $\deg \mathfrak{p}^{n/e} = n$. Therefore the number of integral divisors of degree n in K is at least $\bar{N}_1/n$. On the other hand, the Riemann-Roch theorem says that the dimension of a divisor class C of degree $n = 2g_K - 1$ in K is

$$d(C) = \deg C + 1 - g_K = g_K = d - 1.$$

There are precisely $(q^{d(c)}-1)/(q-1)$ integral divisors in each class C (see [2], p.64), and we have $h(K)$ divisor classes of degree n. Therefore

$$\begin{aligned}\frac{h(K)(q^{d-1}-1)}{q-1} &\geq \frac{\bar{N}_1}{n} \geq \frac{q^n+1-2(d-1)q^{n/2}}{n} \\ &= \frac{q^{2d-3}+1-2(d-1)q^{\frac{2d-3}{2}}}{2d-3},\end{aligned}$$

and

$$(*) \qquad h(O_K) \geq \frac{(q-1)(q^{2d-3}+1-2(d-1)q^{\frac{2d-3}{2}})}{d(2d-3)(q^{d-1}-1)}.$$

A simple calculation shows that the right-hand side of $(*)$ is bigger than one if $q \geq 7$, $d \geq 2$; $q=5$, $d \geq 3$; or $q=3$, $d \geq 5$. This completes the proof of theorem 4.1.

For the case of $d=1$, we have $g_K=0$ and $h(O_K)$ is always one. The following result gives all real quadratic function fields of Chowla type with $h(O_K)=1$ beside the trivial case of $d=1$.

Theorem 4.2. *Suppose that* $k=\mathbb{F}_q(x), 2 \nmid q, K=k(\sqrt{D})$, $D=A^2+a, a \in \mathbb{F}_q^*$, A *is monic polynomial in* $\mathbb{F}_q[x]$ *and* $\deg A = d \geq 2$. *There are precisely following six such fields with* $h(O_K)=1$:

$$\begin{aligned}&q=3, \quad D=A^2+1 \quad \text{with} \quad A=x^3-x\pm 1, \quad x^2+1, \quad x^2\pm x-1,\\ &q=5, \quad D=x^4+2.\end{aligned}$$

Proof. From theorem 4.1 we know that there are only following finite cases to be considered:$q=3, 2 \leq d \leq 4$; and $q=5, d=2$. D has to satisfy the condition (I) or (II) of remark 3.6.

(I) D is irreducible, $D=A^2-a, a \in \mathbb{F}_q^*-(\mathbb{F}_q^*)^2$. In [3] a table of $h(K)$ is presented for all quadratic function fields $K=k(\sqrt{P})$ where P is irreducible and $3 \leq \deg P \leq 8$ for $q=3$; $3 \leq \deg P \leq 5$ for $q=5$, and $3 \leq \deg P \leq 4$ for $q=7$,11. From this table we find exactly six fields $K=k(\sqrt{P})$ mentioned in theorem 4.2 satisfying the condition (I) and $h(O_K)=\frac{h(K)}{d}=1$.

Our class number table was made by using the following analytic formula given by E.Artin [1]:

$$(1) \qquad h(K) = -\sum_{i=1}^{2d-1} i\sigma_i(D),$$

where

$$(2) \qquad \sigma_i(D) = \sum_{\substack{A\in\mathbb{F}_q[x] \\ \text{monic} \\ (A,D)=1 \\ \deg A=i}} \left[\frac{D}{A}\right]$$

and $[\frac{D}{A}]$ is the Jacobi symbol for polynomials in $\mathbb{F}_q[x]$ which is a natural analogy of the ordinary Jacobi symbol and has similar properties:

(1) $(\frac{M_1M_2}{N})=(\frac{M_1}{N})(\frac{M_2}{N})$ if $M_1, M_2 \in \mathbb{F}_q[x], N \in \mathbb{F}_q[x]$ is monic, and $(M_1,N)=(M_2,N)=1$.

(2)$(\frac{a}{N})=a^{\frac{|N|-1}{2}}=a^{\frac{q-1}{2}\deg N}$, if $a \in \mathbb{F}_q^*, N \in \mathbb{F}_q[x]$ is monic and $|N|=q^{\deg N}$. Particularly, $(\frac{q}{N})=(-1)^{\deg N}$.

(3) (Reciprocity law) $(\frac{M}{N})(\frac{N}{M}) = (-1)^{\frac{|M|-1}{2}\cdot\frac{|N|-1}{2}} = (-1)^{\frac{q-1}{2}\deg M\cdot\deg N}$ if M and N are monic polynomials in $\mathbb{F}_q[x]$ and $(M, N) = 1$.

Moreover, we know that $\sigma_0 = 1, \sigma_{2d-1} = -q^{d-1}$ and

$$\sigma_{2d-i} = q^{d-i}[-\sigma_{i-1} + (q-1)(\sigma_{i-2} + \cdots + \sigma_1 + \sigma_0)] \qquad (2 \le i \le d). \tag{3}$$

Therefore we need to compute $\sigma_i(D)$ for $1 \le i \le d-1$ only.

For the condition (II),$D = M^2 - b^2 = (M + b)(M - b), b \in \mathbb{F}_q^*, 2 \nmid d = \deg M \ge 2$ and $M \pm b$ are irreducible. The only case we need to consider is $q = 3$ and $d = 3$. There is only one field: $D = (x^3 - x + 1)(x^3 - x - 1)$. For this field we have $\sigma_0 = 1, \sigma_5 = -9$. The formula (2) gives $\sigma_1 = -3$ and $\sigma_2 = 9$. Then from (3) we have $\sigma_4 = 15$ and $\sigma_3 = -13$. Therefore from the analytic formula (1)

$$h(O_K) = \frac{h(K)}{3} = -\frac{1}{3}(-3 + 18 - 39 + 60 - 45) = 3 \ne 1.$$

This completes the proof of theorem 4.2.

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