PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1473–1482
S 0002-9939(99)05016-9
Article electronically published on January 29, 1999

COMPLETENESS OF EIGENVECTORS OF GROUP REPRESENTATIONS OF OPERATORS WHOSE ARVESON SPECTRUM IS SCATTERED

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(Communicated by David R. Larson)

Abstract. We establish the following result.

Theorem. *Let $\alpha : G \to \mathcal{L}(X)$ be a $\sigma(X, X_*)$ integrable bounded group representation whose Arveson spectrum* $\mathrm{Sp}(\alpha)$ *is scattered. Then the subspace generated by all eigenvectors of the dual representation α^* is w^* dense in X^*. Moreover, the $\sigma(X, X_*)$ closed subalgebra W_α generated by the operators α_t $(t \in G)$ is semisimple.*

If, in addition, X does not contain any copy of c_0, then the subspace spanned by all eigenvectors of α is $\sigma(X, X_*)$ dense in X. Hence, the representation α is almost periodic whenever it is strongly continuous.

1. Spectral theory for integrable bounded group representations

Throughout this paper G will denote a locally compact abelian (LCA) group with identity e and $\widehat{G}$ will denote the dual group of G. The multiplication on LCA groups will be written by addition. Let $L^1(G)$ (resp. $M(G)$) be the usual group algebra (resp. measure algebra) with convolution as product operation. We refer to [11] or [21] for basic knowledge of Harmonic Analysis on LCA groups.

Given a complex Banach space X, let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. Take a LCA group G. A bounded group representation α of G on X is a mapping $\alpha : G \to \mathcal{L}(X)$ satisfying the following properties:

(a) **Group property:** $\alpha_e = I_X$ the identity operator on X and $\alpha_{s+t} = \alpha_s \alpha_t$ for all $s, t \in G$;

(b) **Boundedness:** $\|\alpha\| := \sup_{t \in G} \|\alpha_t\| < \infty$.

Moreover, α is called *strongly* (resp. *weakly*) *continuous* if for each $x \in X$ the mapping $t \mapsto \alpha_t x$ is norm (resp. weakly) continuous. We need a further notion.

Definition 1.1. A bounded group representation $\alpha : G \to \mathcal{L}(X)$ of G on X is called *integrable* if there exists a subspace $X_* \subset X^*$ satisfying the following

Received by the editors September 1, 1997.
1991 *Mathematics Subject Classification.* Primary 47A67, 47A10.
Key words and phrases. Spectrum of group representation, almost periodicity.

requirements:

(i) X_* is a norm determining subspace of X^*, i.e., the following

$$\|x\|_1 := \sup\{|\rho(x)| : \rho \in X_*, \|\rho\| \leq 1\}, \quad x \in X,$$

defines an equivalent norm on X;

(ii) The group representation α is $\sigma(X, X_*)$ continuous and for each $\mu \in M(G)$ there exists an operator $\tilde{\alpha}_\mu \in \mathcal{L}(X)$ such that

$$\rho(\tilde{\alpha}_\mu x) = \int_G \rho(\alpha_t x)\, d\mu(t) \quad \text{for all} \quad (x, \rho) \in X \times X_*.$$

In this case we say that α is $\sigma(X, X_*)$ integrable and the operators $\tilde{\alpha}_\mu$ are written as

$$\tilde{\alpha}_\mu = \sigma - \int_G \alpha_t\, d\mu(t), \quad \mu \in M(G).$$

It is easily verified that the extension $\tilde{\alpha} : M(G) \to \mathcal{L}(X)$ is a bounded algebra homomorphism, i.e.,

$$\tilde{\alpha}_{\mu * \nu} = \tilde{\alpha}_\mu \tilde{\alpha}_\nu, \quad \mu, \nu \in M(G).$$

Moreover, $\alpha_t = \tilde{\alpha}_{\delta_t}$ $(t \in G)$, where δ_t is the Dirac measure at the point t.

In the sequel $\tilde{\alpha}$ will denote the algebra homomorphism which is obtained by integrating a group homomorphism α.

That all weakly continuous bounded group representations are integrable is well-known; see [1].

Let $\alpha : G \to \mathcal{L}(X)$ be an integrable bounded group representation. For $f \in L^1(G)$, let $\tilde{\alpha}_f$ be the image of the measure $d\mu_f(t) := f(t)dt$ under $\tilde{\alpha}$. It is clear that $\tilde{\alpha} : L^1(G) \to \mathcal{L}(X)$ is also a bounded algebra homomorphism. Let $I_\alpha := \{f \in L^1(G) : \tilde{\alpha}_f = 0\}$. The *Arveson spectrum* of α, denoted by $\mathrm{Sp}(\alpha)$, is defined as the hull of I_α, i.e.,

$$\mathrm{Sp}(\alpha) := \{\gamma \in \widehat{G} : \hat{f}(\gamma) = 0 \quad \text{for} \quad \text{all} \quad f \in I_\alpha\}.$$

For $x \in X$, let $I_x := \{f \in L^1(G) : \tilde{\alpha}_f x = 0\}$ and define

$$\mathrm{Sp}_\alpha(x) := \{\gamma \in \widehat{G} : \hat{f}(\gamma) = 0 \quad \text{for all} \quad f \in I_x\}$$

to be the *spectrum of* α *at the point* x. For a closed subset Λ of $\widehat{G}$, define

$$X^\alpha(\Lambda) := \{x \in X : \mathrm{Sp}_\alpha(x) \subseteq \Lambda\}$$

to be the *spectral subspace corresponding to* Λ. A $\gamma \in \widehat{G}$ is called an *eigenvalue* if the eigenspace $\{x \in X : \alpha_t x = \gamma(t)x \ \forall t \in G\}$ is non-trivial. Eigenvectors are defined similarly.

We need the following basic facts established by the author in [14, Chapter I], cf. [1], [3], [4] and [7]. A complete summary of [14] appeared in "Dissertation Summary in Mathematics" **1** (1996), 171-178.

Theorem 1.2. *Assume $X \neq \{0\}$. Let $\alpha : G \to \mathcal{L}(X)$ be a $\sigma(X, X_*)$ integrable bounded group representation. Then,*

(i) The Arveson spectrum $\mathrm{Sp}(\alpha)$ is a non-empty closed subset of $\widehat{G}$. Assume $\gamma \in \widehat{G}$. Then, $\gamma \in \mathrm{Sp}(\alpha)$ if and only if there exists a net (x_i) of norm-one vectors in X such that $\|\alpha_t x_i - \gamma(t)x_i\| \to 0$ $(i \to \infty)$ uniformly for t in every compact subset of G.

(ii) The group representation α is norm continuous if and only if its Arveson spectrum $\mathrm{Sp}(\alpha)$ *is compact.*

(iii) Assume that $\mathrm{Sp}(\alpha)$ *is decomposed into disjoint closed subsets E and F where E is compact. Then there exists a projection $P \in \{\tilde{\alpha}_f : f \in L^1(G)\}$ such that*

$$X^\alpha(E) = PX \quad and \quad X^\alpha(F) = (I_X - P)X.$$

For the subspace representations $\alpha \circ P$ and $\alpha \circ (I_X - P)$ obtained by restricting α in PX and $(I_X - P)$ there holds

$$\mathrm{Sp}(\alpha \circ P) = E \quad and \quad \mathrm{Sp}(\alpha \circ (I_X - P)) = F.$$

(iv) The following spectral mapping theorem holds:

$$\sigma(\alpha_t) = \overline{\{\gamma(t) : \gamma \in \mathrm{Sp}(\alpha)\}} \quad for\ all \quad t \in G.$$

(v) Let $K(G) := \{f \in L^1(G) :\ \hat{f}\ \ has\ compact\ support\}$. Then, the subspace generated by all vectors $\tilde{\alpha}_f x$ ($f \in K(G)$ and $x \in X$) is $\sigma(X, X_*)$ dense in X.

(vi) If $G := \mathbb{Z}$ and $T \in \mathcal{L}(X)$ is the generator of the representation $\alpha : \mathbb{Z} \to \mathcal{L}(X)$, then $\mathrm{Sp}(\alpha) = \sigma(T)$, *where $\sigma(T)$ is the spectrum of the operator T.*

(vii) If $G := \mathbb{R}$ and A is the infinitesimal generator of the one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$, then $\mathrm{Sp}(\alpha) = i\sigma(A)$, *where $\sigma(A)$ is the spectrum of the closed operator A.*

As a remark we point out that a weakly continuous bounded group representation $\alpha : G \to \mathcal{L}(X)$ is in fact strongly continuous. To see this, take $f \in K(G)$ and let $X_f := \overline{\tilde{\alpha}_f X}$. Consider the restriction of α in X_f, denoted by β. To estimate the spectrum of β, let $\gamma \in \widehat{G} \setminus \mathrm{supp}\hat{f}$. Then, by the regularity of $L^1(G)$ there exists $g \in L^1(G)$ such that $\hat{g}(\gamma) = 1$ and $\mathrm{supp}\hat{g} \subseteq \widehat{G} \setminus \mathrm{supp}\hat{f}$. It follows that $g * f = 0$ and thus $\tilde{\beta}_g\tilde{\alpha}_f = \tilde{\alpha}_g\tilde{\alpha}_f = \tilde{\alpha}_{g*f} = 0$. This implies by definition of $\mathrm{Sp}(\beta)$ that $\gamma \notin \mathrm{Sp}(\beta)$ and thus $\mathrm{Sp}(\beta)$ is contained in the compact subset $\mathrm{supp}\hat{f}$. By Theorem 1.2 (ii) β is norm continuous. This implies that the function $t \mapsto \alpha_t x$ is norm continuous for each $x \in X$ which can be written as $x = \tilde{\alpha}_f y$ for some $y \in X$ and $f \in K(G)$. As claimed by Theorem 1.2 (v), such vectors generate a weakly dense and hence norm dense subspace of X. In conclusion, α is strongly continuous.

We need also two auxiliary results.

Proposition 1.3. *Let $\alpha : G \to \mathcal{L}(X)$ be an integrable bounded group representation. Assume γ to be an isolated point of* $\mathrm{Sp}(\alpha)$. *Then, there exists a projection $0 \neq P_\gamma \in \{\tilde{\alpha}_f : f \in L^1(G)\}$ such that*

$$\alpha_t P_\gamma = \gamma(t) P_\gamma \quad for\ all \quad t \in G.$$

In particular, γ is an eigenvalue of α.

Proof. Let P_γ be the spectral projection corresponding to the set $\{\gamma\}$ for which we have

$$\mathrm{Sp}(\alpha \circ P_\gamma) = \{\gamma\}.$$

The existence of P_γ is guaranteed by Theorem 1.2 (iii). Applying Theorem 1.2 (iv) to the group representation $\alpha \circ P_\gamma$ we find that $\sigma((\alpha \circ P_\gamma)_t) = \{\gamma(t)\}$ for all $t \in G$. It follows from Gelfand's theorem (see [10] or [13]) that $(\alpha \circ P_\gamma)_t = \gamma_0(t) I_{P_\gamma X}$ for all $t \in G$. Clearly, $P_\gamma X \neq \{0\}$. □

The following "lifting property" should be compared with a similar result in [15].

Proposition 1.4. *Let $\alpha : M(G) \to \mathcal{L}(X)$ be a bounded group representation. Let $Y \subseteq Z$ be two $\alpha-$invariant closed subspaces of X. Assume that $\gamma_0 \in \mathrm{Sp}(\alpha)$ and $\psi \in (Z/Y)^*$ satisfy*

$$\langle \psi, \alpha_t(z) + Y\rangle = \gamma_0(t)\langle \psi, z + Y\rangle \quad for \quad all \quad t \in G,\ z \in Z.$$

Then, there exists $x_0^* \in X^*$ such that $\alpha_t^* x_0^* = \gamma_0(t)x_0^*$ for all $t \in G$ and $\langle x_0^*, z\rangle = \langle \psi, z + Y\rangle \quad for \quad all \quad z \in Z$.

Proof. Define $\psi_1 \in Z^*$ by $\psi_1(z) := \psi(z + Y)$, $z \in Z$. By Hahn-Banach theorem we extend ψ_1 to an element $\psi_2 \in X^*$, such that $\|\psi_2\| = \|\psi_1\|$. Since G is abelian, there exists an invariant mean ϕ on $l^\infty(G)$. For each $x \in X$, define

$$F(x; t) := \gamma_0(-t)\psi_2(\alpha_t x), \quad t \in G.$$

This is a function in $l^\infty(G)$. It follows that

$$x_0^*(x) := \phi(F(x; \cdot)), \quad x \in X,$$

well defines a linear functional on X with $\|x_0^*\| \le \|\alpha\| \cdot \|\psi\|$. For $s, t \in G$ and $x \in X$ we have

$$F(\alpha_s(x); t) = \gamma_0(-t)\psi_2(\alpha_{s+t}x) = \gamma_0(s)F(x; s + t).$$

It follows that

$$x_0^*(\alpha_s x) = \gamma_0(s)\phi(F(x; \cdot + s)) = \gamma_0(s)x_0^*(x),$$

where for the last identity we use the translation-invariance of ϕ. Therefore, $\alpha_s^* x_0^* = \gamma_0(s)x_0^*$ for all $s \in G$. Consider $z \in Z$. Then,

$$F(z; t) = \gamma_0(-t)\psi(\alpha_t(z) + Y) = \gamma_0(-t)\gamma_0(t)\psi(z + Y) = \psi(z + Y).$$

This implies that $x_0^*(z) = \phi(F(z; \cdot)) = \psi(z + Y)\phi(\mathbf{1}) = \psi(z + Y)$, completing the proof. ☐

2. Completeness of eigenvectors of dual representations

Recall that a closed subset Λ of $\widehat{G}$ is called *scattered* if each closed subset of Λ contains an isolated point. Every closed countable subset of $\widehat{G}$ is scattered and, moreover, if $\widehat{G}$ satisfies the second axiom of countability, then a closed subset of $\widehat{G}$ is scattered if and only if it is countable.

Our main result in this section reads as follows.

Theorem 2.1. *Let $\alpha : G \to \mathcal{L}(X)$ be an integrable bounded group representation whose Arveson spectrum* $\mathrm{Sp}(\alpha)$ *is scattered. Then the subspace*

$$\begin{aligned} X_{\alpha^*} := \mathrm{lin}\{x^* \in X^* : \quad & there\ exists\ \gamma \in \mathrm{Sp}(\alpha)\ such\ that \\ & \alpha_t^* x^* = \gamma(t)x^* \quad for \quad all \quad t \in G\} \end{aligned}$$

generated by all eigenvectors of α^* is w^*-dense in X^*. Moreover, there exists a uniformly bounded, mutually orthogonal system of projections $\{E_\gamma : \gamma \in \mathrm{Sp}(\alpha)\}$ such that $E_\gamma X^* = \{x^* \in X^* : \alpha_t^* x^* = \gamma(t)x^*\ for\ all\ t \in G\}$ for all $\gamma \in \mathrm{Sp}(\alpha)$.

Proof. Let $Y := \{y \in X : \ x^*(y) = 0 \ \text{ for all } \ x^* \in X_{\alpha^*}\}$. Then, Y is an $\alpha-$invariant closed subspace of X. We have to show $Y = \{0\}$. To this end, let β_t $(t \in G)$ be the restriction of α_t in Y. Then, β is also an integrable bounded group representation. By Theorem 1.2 (i), it is sufficient to prove that $\mathrm{Sp}(\beta) = \emptyset$ in order to obtain $Y = \{0\}$.

Assume conversely that $Y \neq \{0\}$. Let $X_* \subset X^*$ be such that α is $\sigma(X, X_*)$ integrable. Let $Y_* := \{x^*_{|Y} : x^* \in Y\}$. Then, β is $\sigma(Y, Y_*)$ integrable. Let V be the norm closure of Y_* in Y^*. Then, β is also integrable with respect to the weak topology $\sigma(Y, V)$. We have $V \neq \{0\}$ and $\beta_t^* V \subset V$ for each $t \in G$. This implies that

$$\Phi_t := \text{the restriction of } \ \beta_t^* \ \text{ in } \ V, \quad t \in G,$$

defines a $\sigma(V, Y)$ integrable bounded group representation. To compute the spectrum of Φ, consider $f \in I_\alpha$, i.e., $\tilde{\alpha}_f = 0$. Then, for all $y \in Y$ and $\rho \in Y_*$ we have

$$\langle y, \tilde{\Phi}_f \rho\rangle = \int_G \langle y, \Phi_t \rho\rangle f(t)\, dt = \int_G \langle \rho, \alpha_t y\rangle f(t)\, dt = \rho(\tilde{\alpha}_f y) = 0.$$

Therefore, $\tilde{\Phi}_f = 0$ and thus $I_\Phi \supseteq I_\alpha$. By definition of spectrum we find that $\mathrm{Sp}(\Phi) \subseteq \mathrm{Sp}(\alpha)$. Since $\mathrm{Sp}(\alpha)$ is scattered, so is $\mathrm{Sp}(\Phi)$. The non-empty scattered set $\mathrm{Sp}(\Phi)$ contains an isolated point, γ_0 say. By Proposition 1.3 γ_0 is an eigenvalue of Φ. Choose $0 \neq y_0^* \in V \subseteq Y^*$ to be an eigenvector for γ_0. Then, for all $y \in Y$ and $t \in G$ we have

$$y_0^*(\alpha_t y) = \Phi_t y_0^*(y) = \gamma_0(t) y_0^*(y).$$

Applying Proposition 1.4 to α with the case $Z = \{0\}$ we obtain an extension $0 \neq x_0^* \in X^*$ of y_0^* such that $\alpha_t^* x_0^* = \gamma_0(t) x_0^*$ for all $t \in G$. It follows that $x_0^* \in X_{\alpha^*}$. Hence, we have for all $y \in Y$ that $y_0^*(y) = x_0^*(y) = 0$, a contradiction.

To show the "Moreover" part, let ϕ be an invariant mean on $l^\infty(G)$. Let $\gamma \in \mathrm{Sp}(\alpha)$. For each pair $(x, \rho) \in X \times X^*$ the function $t \mapsto \langle \rho, \gamma(-t)\alpha_t x\rangle$ belongs to $l^\infty(G)$. Thus,

$$\langle E_\gamma \rho, x\rangle := \phi_t(\langle \rho, \gamma(-t)\alpha_t x\rangle, \ x \in X, \ \rho \in X^*,$$

defines an operator E_γ on X^*, where ϕ_t means that the invariant mean is applied to the corresponding function of t. It is evident that $\|E_\gamma\| \leq \|\alpha\|$. The translation-invariance of ϕ implies that $\alpha_t^* E_\gamma = \gamma(t) E_\gamma$ for all $t \in G$. From this we see further that each E_γ is a projection. To prove the mutual orthogonality, let $\gamma_1, \gamma_2 \in \mathrm{Sp}(\alpha)$ be two different elements. Clearly, E_{γ_1} and E_{γ_2} are commuting. Therefore, for all $t \in G$

$$\begin{aligned} \gamma_1(t) E_{\gamma_1} E_{\gamma_2} &= \alpha_t^* E_{\gamma_1} E_{\gamma_2} = \alpha_t^* E_{\gamma_2} E_{\gamma_1} \\ &= \gamma_2(t) E_{\gamma_2} E_{\gamma_1} = \gamma_2(t) E_{\gamma_1} E_{\gamma_2}. \end{aligned}$$

Since $\gamma_1 \neq \gamma_2$, this implies that $E_{\gamma_1} E_{\gamma_2} = 0$. The proof is finished. □

Consider a representation α given as in Theorem 2.1 which is $\sigma(X, X_*)$ integrable. Let W_α be the $\sigma(X, X_*)$ closed subalgebra of $\mathcal{L}(X)$ generated by all operators $\tilde{\alpha}_\mu$ $(\mu \in M(G))$. Assume $T \in W_\alpha$. Then, there exists a net $(\mu_i) \subset M(G)$ such that $\tilde{\alpha}_{\mu_i} \xrightarrow{\sigma} T$. For $\gamma \in \mathrm{Sp}(\alpha)$ consider the projection E_γ given in Theorem 2.1. Fix

$(x, \rho) \in X \times X_*$. Then, from the definition of $\tilde{\alpha}_{\mu_i}$ we have

$$\begin{aligned} \langle E_\gamma \rho, \tilde{\alpha}_{\mu_i} x \rangle &= \int_G \langle \alpha_t^* E_\gamma \rho, x \rangle \, d\mu_i(t) \\ &= \int_G \langle \gamma(t) E_\gamma \rho, x \rangle \, d\mu_i(t) = \widehat{\mu_i}(\gamma) \langle E_\gamma \rho, x \rangle. \end{aligned}$$

By taking limit we find

$$\langle E_\gamma \rho, Tx \rangle = \lim_i \widehat{\mu_i}(\gamma) \langle E_\gamma \rho, x \rangle.$$

Since this identity holds for all $x \in X$ and $\rho \in X_*$, it follows that the limit $\lim_i \widehat{\mu_i}(\gamma)$ exists; denoted by c_γ. Then we have $T^* E_\gamma = c_\gamma E_\gamma$. Therefore, the subspace generated by all eigenvectors of T^* is w^*-dense in X^*. As a consequence, the zero operator is the unique nilpotent operator in W_α and thus W_α is a semisimple Banach algebra.

This has established the following Theorem 2.2 which refines [20, Theorem 7] where the corresponding result is given for weakly continuous representations. Note that our proof is completely different from that of [20].

Theorem 2.2. *Let $\alpha : G \to \mathcal{L}(X)$ be a $\sigma(X, X_*)$ integrable bounded group representation whose Arveson spectrum $\mathrm{Sp}(\alpha)$ is scattered. Then the $\sigma(X, X_*)$ closed subalgebra W_α of $\mathcal{L}(X)$ is semisimple.*

Moreover, for each $T \in W_\alpha$ there exists a set $\{c_\gamma : \gamma \in \mathrm{Sp}(\alpha)\} \subset \mathbb{C}$ such that $T^* E_\gamma = c_\gamma E_\gamma$ for all $\gamma \in \mathrm{Sp}(\alpha)$, where $\{E_\gamma : \gamma \in \mathrm{Sp}(\alpha)\}$ is the set of projections given in Theorem 2.1. As a result, the subspace generated by all eigenvectors of T^* is w^*-dense in X^*.

As consequences of Theorems 2.1 and 2.2 we have:

Corollary 2.3. *Let $T \in \mathcal{L}(X)$ be a doubly power bounded operator with countable spectrum $\sigma(T)$. Then the subspace generated by all eigenvectors of T^* is w^*-dense in X^*. Moreover, the weakly closed subalgebra generated by T is semisimple.*

Proof. Let $\alpha : \mathbb{Z} \to \mathcal{L}(X)$ be the group representation given by

$$\alpha_n := T^n \quad \text{for all} \quad n \in \mathbb{Z}.$$

α is norm continuous and bounded. Moreover, by Theorem 1.2 (vi) we have $\mathrm{Sp}(\alpha) = \sigma(T)$. Hence, Theorems 2.1 and 2.2 are applicable to α and yields the desired result. □

Corollary 2.4. *Let $(e^{tA})_{t \in \mathbb{R}}$ be a strongly continuous bounded group of operators on X such that the spectrum $\sigma(A)$ of the infinitesimal generator A is countable. Then the subspace generated by all eigenvectors of the dual operator A^* is w^*-dense in X^*. Moreover, the smallest weakly closed subalgebra of $\mathcal{L}(X)$ containing all operators e^{tA} $(t \in \mathbb{R})$ is semisimple.*

Proof. Let $\alpha : \mathbb{R} \to \mathcal{L}(X)$ be the strongly continuous bounded group representation given by

$$\alpha_t := e^{tA} \quad \text{for all} \quad t \in \mathbb{R}.$$

Then $\mathrm{Sp}(\alpha) = i\sigma(A)$ by Theorem 1.2 (vii). Hence $\mathrm{Sp}(\alpha)$ is countable. If $\lambda \in \sigma(A)$ and $x^* \in X^*$ satisfy $\alpha_t^* x^* = e^{\lambda t} x^*$ for all $t \in \mathbb{R}$, then x^* belongs to the definition domain of A^* and $A^* x^* = \lambda x^*$. Thus, the result follows by using Theorems 2.1 and 2.2. □

Recall that an operator $T \in \mathcal{L}(X)$ is called *hermitian* if $\|e^{itT}\| = 1$ for all $t \in \mathbb{R}$. From Corollary 2.4 we derive immediately the following analogue of Corollary 2.3.

Corollary 2.5. *Let $T \in \mathcal{L}(X)$ be a hermitian operator with countable spectrum $\sigma(T)$. Then the subspace generated by all eigenvectors of T^* is w^*-dense in X^*. Moreover, the weakly closed subalgebra generated by T is semisimple.*

We remark that the semisimplicity in Corollary 2.3 and Corollary 2.5 has been proved by Feldman [9] and Sinclair [22, Theorem 3.1], respectively. Their methods are completely different from that of [20] and ours.

3. Completeness of eigenvectors

The completeness of eigenvectors of integrable group representations with discrete spectrum is guaranteed by the following result.

Theorem 3.1. *Let $\alpha : G \to \mathcal{L}(X)$ be an $\sigma(X, X_*)$ integrable bounded representation with discrete Arveson spectrum. Then the subspace*

$$X_\alpha := \mathrm{lin}\{x \in X : \quad \textit{there exists } \gamma \in \mathrm{Sp}(\alpha) \textit{ such that } \alpha_t x = \gamma(t)x \quad \textit{for all} \quad t \in G\}$$

generated by all eigenvectors of α is $\sigma(X, X_*)$ dense in X. Moreover, there exists a uniformly bounded, mutually orthogonal system of projections $\{P_\gamma : \gamma \in \mathrm{Sp}(\alpha)\}$ such that $P_\gamma X = \{x \in X : \alpha_t x = \gamma(t)x$ for all $t \in G\}$ for all $\gamma \in \mathrm{Sp}(\alpha)$.

Proof. Applying Proposition 1.3 to α we find that for each $\gamma \in \mathrm{Sp}(\alpha)$ there exists a projection $P_\gamma \in \{\tilde{\alpha}_f : f \in L^1(G)\}$ satisfying

$$\alpha_t P_\gamma = \gamma(t) P_\gamma \quad \text{for all} \quad t \in G.$$

Thus,

$$X_\alpha = \bigvee_{\gamma \in \mathrm{Sp}(\alpha)} P_\gamma X.$$

Consider $f \in K(G)$, i.e., $\hat{f}$ has compact support. Let $X_f := \overline{\tilde{\alpha}_f X}$ and β be the restriction of α in X_f. Then, β is integrable and

$$\mathrm{Sp}(\beta) \subseteq \mathrm{Sp}(\alpha) \cap \mathrm{supp}\hat{f}.$$

Note that the set $\mathrm{Sp}(\alpha) \cap \mathrm{supp}\hat{f}$ is compact and discrete; hence it contains at most finitely many points. Therefore, $\mathrm{Sp}(\beta)$ is a finite subset of $\mathrm{Sp}(\alpha)$. It follows from Theorem 1.2 (iii) combining with Proposition 1.3 that X_f is decomposed into finite sum of eigenspaces of β. Thus, X_f is contained in X_α. By Theorem 1.2 (v), the subspace generated by all of the subsets X_f ($f \in K(G)$) is $\sigma(X, X_*)$ dense in X. Consequently, X_α is $\sigma(X, X_*)$ dense in X. □

A more general result of Theorem 3.1 is given in [14, Theorem 3.1.2].

In what follows we will study the almost periodicity of representations. Let $BUC(G; X)$ be the Banach space of all bounded and uniformly continuous functions $h : G \to X$ with the norm

$$\|h\| := \sup_{t \in G} \|h(t)\|_X.$$

For each $t \in G$ let T_t be the translation on $BUC(G; X)$ given by

$$T_t h(\cdot) := h(\cdot + t) \quad \text{for all} \quad h \in BUC(G; X).$$

A function $h \in BUC(G;X)$ is called *almost periodic* if the set $\{T_t h : t \in G\}$ is relatively compact in $BUC(G;X)$. Let $AP(G;X)$ be the subspace of all almost periodic functions in $BUC(G;X)$ and denote by $AP(G)$ the space $AP(G;\mathbb{C})$. A function $h \in BUC(G;X)$ is called *scalar almost periodic* if for each $x^* \in X^*$ the scalar function $t \mapsto x^*(h(t))$ is almost periodic. Let $h \in BUC(G;X)$. It is well known that $h \in AP(G;X)$ if and only if h is scalar almost periodic and the range $\{h(t) : t \in G\}$ is relatively compact in X (cf. [19, pp. 70-72]).

The *Beurling spectrum* of $h \in BUC(G;X)$, denoted by $\sigma(h)$, is defined to be the local spectrum of T at h, i.e.,

$$\sigma(h) := \mathrm{Sp}_T(h) = \{\gamma \in \widehat{G} : f \in L^1(G), \tilde{T}_f h = 0 \Longrightarrow \hat{f}(\gamma) = 0\}.$$

If $h \in BUC(G;\mathbb{C})$ has scattered spectrum, then a theorem of Loomis [17, Theorem 5] asserts that h is almost periodic. The extension of Loomis's theorem to vector-valued functions is given by Baskakov [2, Theorem 2] as follows. Recall that c_0 is the Banach space of all convergent sequences $(a_n)_{n\in\mathbb{N}}$ with limit zero.

Theorem 3.2. *Assume that $h \in BUC(G;X)$ has scattered spectrum. If X does not contain any copy of c_0, then h is almost periodic.*

We call a strongly continuous bounded group representation $\alpha : G \to \mathcal{L}(X)$ *almost periodic* if for each $x \in X$ the function $t \mapsto \alpha_t x$ is almost periodic. It follows from the Jocob-deLeeuw-Glicksberg theory that the subspace generated by all eigenvectors of an almost periodic representation is norm dense in the defining Banach space, see [6, Theorem 4.11], [16] or the Basic Theorem in [18, p.150].

Let $\alpha : G \to \mathcal{L}(X)$ be a $\sigma(X, X_*)$ integrable bounded group representation whose Arveson spectrum is scattered. Assume further that the Banach space X does not contain any copy of c_0. For $f \in K(G)$, let $X_f := \overline{\tilde{\alpha}_f X}$ and let β be the restriction of α in X_f. As seen in the proof of Theorem 3.1, we have

$$\mathrm{Sp}(\beta) \subseteq \mathrm{Sp}(\alpha) \cap \mathrm{supp}\hat{f}.$$

Hence, the representation β has compact spectrum and thus is norm continuous by Theorem 1.2 (ii). Given $x_0 \in X_f$, let

$$h(t) := \beta_t x_0 = \alpha_t x_0, \quad t \in G.$$

Then, the norm continuity of β implies that $h \in BUC(G;X_f)$. Let $g \in L^1(G)$. For $s \in G$ we have

$$\tilde{T}_g h(s) = \int_G T_t h(s) g(t)\, dt = \int_G h(s+t) g(t)\, dt = \int_G \alpha_{s+t} x_0 g(t)\, dt = \alpha_s \tilde{\alpha}_g x_0.$$

It follows that $\tilde{T}_g h = 0 \Longleftrightarrow \tilde{\alpha}_g x_0$. By definition we find

$$\sigma(h) = \mathrm{Sp}_\alpha(x_0) \subseteq \mathrm{Sp}(\alpha).$$

Therefore, h has scattered spectrum. It follows from Theorem 3.1 that h is almost periodic. By definition, β is almost periodic and thus by the Jacob-deLeeuw-Glicksberg theory [6] the subspace generated by all eigenvectors of β is norm dense in X_f. Note that the subspace generated by the subspaces X_f ($f \in K(G)$) is $\sigma(X, X_*)$ dense in X by Theorem 1.2 (v). We have proved the following result.

Theorem 3.3. *Let $\alpha : G \to \mathcal{L}(X)$ be a $\sigma(X, X_*)$ integrable bounded group representation whose Arveson spectrum* $\mathrm{Sp}(\alpha)$ *is scattered. Assume that X does not* *contain any copy of* c_0. *Then the subspace*

$$X_\alpha := \operatorname{lin}\{x \in X : \quad \text{there exists } \gamma \in \operatorname{Sp}(\alpha) \text{ such that } \alpha_t x = \gamma(t)x \quad \text{for all} \quad t \in G\}$$

generated by all eigenvectors of α *is* $\sigma(X, X_*)$ *dense in* X.

As a result, if α *is strongly continuous, then* α *is almost periodic.*

The following refines Corollaries 2.3, 2.4 and 2.5.

Corollary 3.4. *Assume that* X *does not contain any copy of* c_0. *Then*

(i) If $T \in \mathcal{L}(X)$ *is a doubly power bounded operator with countable spectrum* $\sigma(T)$, *then the subspace generated by all eigenvectors of* T *is norm dense in* X.

(ii) If $(e^{tA})_{t\in\mathbb{R}}$ *is a strongly continuous bounded group of operators on* X *such that the spectrum* $\sigma(A)$ *is countable, then the subspace generated by all eigenvectors of* A *is norm dense in* X.

(iii) If $T \in \mathcal{L}(X)$ *is a hermitian operator with countable spectrum* $\sigma(T)$, *then the subspace generated by all eigenvectors of* T *is norm dense in* X.

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